Math 2177
Autumn 2018
Midterm 1
09/28/18
Time Limit: 55 Minutes

Name: _____

This exam contains 8 pages (including this cover page) and 5 problems. Write your name on the top of this page.

YOU MAY *NOT* USE YOUR BOOKS, NOTES, OR ANY CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

If you need more space, use the back of the pages or extra page at the end of the exam; clearly indicate when you have done this. Do not rip off pages from the exam.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	15	
3	20	
4	15	
5	30	
Total:	100	

1. (20 points) Consider the function $f(x,y) = x^2 + \frac{1}{2}(y^2 + y)$.

   (a) (5 points) Find critical points of the given $f(x,y)$ and classify them. Compute the values of f at the critical points.

Solution:

$$\begin{cases} f_x(x,y) = 0, \\ f_y(x,y) = 0 \end{cases} \longrightarrow \begin{cases} 2x = 0, \\ y + \dfrac{1}{2} = 0 \end{cases} \longrightarrow \begin{cases} x = 0, \\ y = -\dfrac{1}{2} \end{cases}$$

The point $(0, -\frac{1}{2})$ is the only critical point of f.
Classify the type: We compute

$$f_{xx}(x,y) = 2, \quad f_{yy}(x,y) = 1, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0.$$

Therefore, the discriminant is equal to

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 2 \cdot 1 - 0^2 = 2.$$

In particular, $D(0, -\frac{1}{2}) = 2 > 0$. Moreover, $f_{xx}(0, -\frac{1}{2}) = 2 > 0$. Therefore, f has a local minimum value at $(0, -\frac{1}{2})$.
We have

$$f(0, -\frac{1}{2}) = 0^2 + \frac{1}{2}\left(\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)\right) = -\frac{1}{8}.$$

   (b) (10 points) Use the method of Lagrange multipliers to find the maximum and the minimum values of the given $f(x,y)$ on the unit circle $x^2 + y^2 = 1$.

Solution: Let $g(x,y) = x^2 + y^2 - 1$. We need to find the maximum and the minimum values of the given $f(x,y)$ when $g(x,y) = 0$. Notice that f, g are differentiable and $\nabla g =< 2x, 2y >\neq 0$ on $g(x,y) = 0$ because $\nabla g =< 0, 0 >$ at $(0,0)$ but $g(0,0) = 0^2 + 0^2 - 1 = -1 \neq 0$.
Use Lagrange multipliers to find the maximum and the minimum values of $f(x,y)$ on $g(x,y) = 0$. We solve the following system to find extremal points.

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y), \\ f_y(x,y) = \lambda g_y(x,y), \\ g(x,y) = 0 \end{cases} \longrightarrow \begin{cases} 2x = \lambda \cdot 2x, \\ y + \dfrac{1}{2} = \lambda \cdot 2y, \\ x^2 + y^2 = 1 \end{cases} \longrightarrow \begin{cases} 2x(1-\lambda) = 0, \\ y + \dfrac{1}{2} = \lambda \cdot 2y, \\ x^2 + y^2 = 1 \end{cases}$$

To have $2x(1-\lambda) = 0$, we have that either $x = 0$ or $\lambda = 1$. Therefore, we need to solve two systems:

$$\begin{cases} x = 0, \\ y + \dfrac{1}{2} = \lambda \cdot 2y, \\ x^2 + y^2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} \lambda = 1, \\ y + \dfrac{1}{2} = \lambda \cdot 2y, \\ x^2 + y^2 = 1 \end{cases}$$

The first system gives us solutions $(x, y, \lambda) = (0, 1, \frac{3}{4})$ and $(x, y, \lambda) = (0, -1, -\frac{1}{4})$. The second system gives us solutions $(x, y, \lambda) = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 1)$ and $(x, y, \lambda) = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1)$. Therefore, we get extremal points $(0, 1)$, $(0, -1)$, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$. Now we compute the values of $f(x, y)$ at these extremal points.

$$f(0,1) = 1, \qquad f(0,-1) = 0, \qquad f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{9}{8}, \qquad \text{and} \qquad f(-\frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{9}{8}. \quad (1)$$

Therefore, the minimum values of f on $x^2 + y^2 = 1$ is 0 and the maximum value of f on $x^2 + y^2 = 1$ is $\frac{9}{8}$.

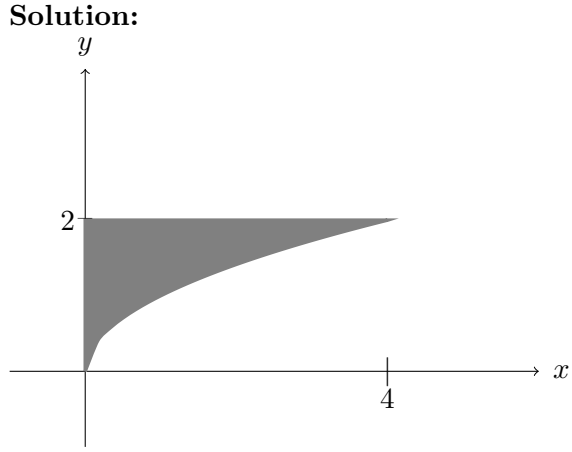
(c) (5 points) Find the absolute maximum and the absolute minimum values of the given $f(x, y)$ on the unit disk $x^2 + y^2 \leq 1$. Use parts (a) and (b).

Solution: In (a) we found the critical point $(0, -\frac{1}{2})$, which belongs to $x^2 + y^2 \leq 1$ because $0^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{4} < 1$. Therefore, by (a) and (b), the absolute maximum of $f(x, y)$ on the unit disk $x^2 + y^2 \leq 1$ is $\frac{9}{8}$ and the absolute minimum of $f(x, y)$ on the unit disk $x^2 + y^2 \leq 1$ is $-\frac{1}{8}$.

2. (15 points) Consider the integral $\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\,dy\,dx$.

(a) (5 points) Draw a picture of the region of integration for the given integral.

Solution:



(b) (5 points) Exchange the order of integration to express the given integral in the order $dx\,dy$.

Solution: Using the picture from part (a), we have

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\,dy\,dx = \int_0^2 \int_0^{y^2} e^{y^3}\,dx\,dy$$

(c) (5 points) Evaluate the integral you got in the previous item (i.e., after exchange of the order of integration).

Solution: We compute

$$\int_0^2 \int_0^{y^2} e^{y^3}\,dx\,dy = \int_0^2 [x]_0^{y^2} e^{y^3}\,dy = \int_0^2 y^2 e^{y^3}\,dy = \int_0^8 \frac{1}{3} e^u du = \frac{1}{3}(e^8 - 1)$$

using the substitution $u = y^3$.

3. (a) (10 points) Rewrite the iterated integral $\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} \left(\frac{y}{x}\right)^3 dx\, dy$ as an iterated integral in polar coordinates. Give the integrand and bounds, but DO NOT EVALUATE.

Solution: Using $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} \left(\frac{y}{x}\right)^3 dx\, dy = \int_0^{\frac{\pi}{4}} \int_0^1 r\tan^3\theta\, dr\, d\theta$$

(b) (10 points) Rewrite the integral from part (a) as an iterated integral involving the variables $u = x^2 + y^2$ and $v = \frac{y}{x}$. Give the integrand and bounds, but DO NOT EVALUATE.
<u>Hint:</u> Consider the origin as a boundary curve described by the equation $x^2 + y^2 = 0$.

Solution: We have

$$\left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 2x & 2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}\right| = \left(2 + 2\frac{y^2}{x^2}\right) = (2 + 2v^2).$$

Thus, we have

$$dx\, dy = \frac{1}{2(1+v^2)}\, du\, dv.$$

The original region is bounded by the curves $y = 0, y = x$ and $x^2 + y^2 = 1$. In the new coordinates, these become $v = 0, v = 1$, and $u = 1$. However, we also get the boundary $u = 0$ from the origin. Therefore,

$$\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} \left(\frac{y}{x}\right)^3 dx\, dy = \int_0^1 \int_0^1 \frac{v^3}{2(1+v^2)}\, du\, dv.$$

4. (15 points) Set up an integral for the volume of the region bounded by $z = \sqrt{3(x^2+y^2)}$ and $x^2+y^2+z^2 = 16$

   (a) (5 points) in Euclidean coordinates. DO NOT EVALUATE!

   Solution: The cone and the sphere intersect in a curve where $x^2+y^2+z^2 = 4(x^2+y^2) = 16$ so $x^2+y^2 = 4$ and $z = 2\sqrt{3}$. Thus, the curve is a circle of radius 2 in the plane $z = 2\sqrt{3}$. The integral is then

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{16-(x^2+y^2)}} 1 \, dz \, dx \, dy.$$

   (b) (5 points) in cylindrical coordinates. DO NOT EVALUATE!

   Solution: The same picture applies and is easier to describe in cylindrical coordinates as $r^2 = x^2+y^2$. The integral is

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{\sqrt{3}r}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta.$$

   (c) (5 points) in spherical coordinates. DO NOT EVALUATE!

   <u>Hint</u>: $\cos(\frac{\pi}{3}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\cos(\frac{\pi}{6}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

   Solution: In spherical coordinates, the bounds are even easier. The sphere is given by $\rho = 4$. The angle of the cone is $\varphi = \pi/6$ from $z = 4\cos\varphi = 2\sqrt{3}$ on the intersection circle. Thus, the integral is

$$\int_{0}^{\pi/6} \int_{0}^{2\pi} \int_{0}^{4} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

5. (30 points) Let $\bar{F}(x,y) = 6x^3y^2\bar{i} + 3x^4y\bar{j}$.

(a) (10 points) Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ that describes the work of $\bar{F}$ along C, where C is the line segment from $(0,0)$ to $(1,1)$.

Solution: We parameterize C as $\bar{r}(t) = (t,t)$ for $t = 0$ to $t = 1$. Therefore, $\bar{F}(\bar{r}(t)) = (6t^5, 3t^5)$ and $d\bar{r} = (1,1)\,dt$. We then compute

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (6t^5, 3t^5)\cdot(1,1)\,dt = \int_0^1 9t^5\,dt = \frac{3}{2}.$$

(b) (5 points) Show that $\bar{F}$ is a conservative vector field.

Solution: We have

$$\frac{\partial P}{\partial y} = 12x^3y = \frac{\partial Q}{\partial x}.$$

Since $\bar{F}$ is defined on all of $\mathbb{R}^2$, this is enough to deduce that $\bar{F}$ is conservative.

(c) (10 points) Find a potential for $\bar{F}$.

Solution: We want a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. Integrating Q with respect to y, we get

$$f(x,y) = \frac{3}{2}x^4y^2 + g(x).$$

Differentiating with respect to x gives

$$P = 6x^3y^2 = 6x^3y^2 + g'(x)$$

so $g'(x) = 0$. Thus, $g(x)$ is constant and we can take it to be zero as we are just looking for one potential. We get

$$f(x,y) = \frac{3}{2}x^4y^2.$$

(d) (5 points) Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ that describes the work of $\bar{F}$ along C, where C consists of the curve $y = xe^{x^2-1} + \sin(\pi x)$ from $(0,0)$ to $(1,1)$. Use parts (b) and (c).

Solution: Using part (c), we can apply the fundamental theorem of line integrals to get

$$\int_C \bar{F}\cdot d\bar{r} = f(1,1) - f(0,0) = \frac{3}{2}.$$

Alternatively, we know that $\bar{F}$ is conservative so the integral is independent of the path from $(0,0)$ to $(1,1)$ implying that the answer should coincide with that of (a).

(EXTRA PAGE 1)