(6)

Lecture 7

Change of variables in $\iint$

Recall $\displaystyle\int_0^1 3\sqrt{3x+1}\,dx = \int_1^4 \sqrt{u}\,du$

$u = 3x+1$

$du = 3dx$

If $x = 0 \to u = 1$

If $x = 1 \to u = 4$

Can we do similar in $\iint$? Yes

$$\iint_R f(x,y)\,dA \overset{\text{want}}{=} \iint_{\tilde{R}} g(u,v)\,du\,dv$$

domain in (u,v)

Let u, v be functions of x and y

Def The Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x \cdot v_y - u_y \cdot v_x$

Then!

$$dx\,dy = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}\,du\,dv$$

absolute value

The other way

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| du\,dv$$

$$= \left| x_u \cdot y_v - y_u \cdot x_v \right| du\,dv$$

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We have

$$\iint_R f(x,y)\,dxdy = \iint_{\tilde R} f(x(u,v),y(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}^{dxdy} =$$

$$= \iint_{\tilde R} f(x(u,v),y(u,v)) \overbrace{\underbrace{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}}_{\substack{\text{in final expression}\\ \text{change } x \to x(u,v)\\ y \to y(u,v)}} dudv}^{dx\,dy}$$

Example Compute $\int_0^1\int_0^1 x^2y\,dxdy$ by setting

$u=x,\ v=xy$

Solution:

1) $dxdy = \ ?$

Jacobian: $\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = |x| = x$$

absolute value

here as $x \geq 0$

want in terms of u, v only

$$dxdy = x^{-1}dudv = \frac{1}{x}dudv = \frac{1}{u}dudv$$

as $x=u$

2) Express f in terms of u and v.
(Need $x = x(u,v)$ and $y = y(u,v)$)

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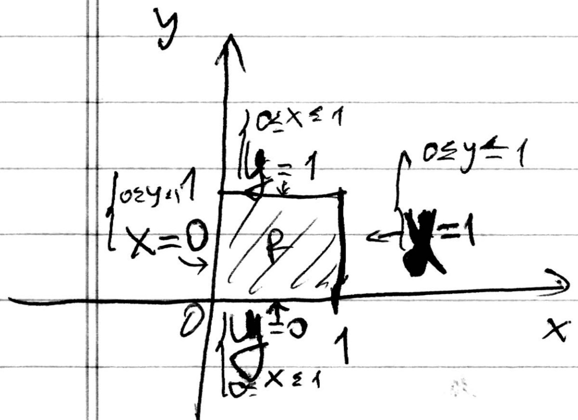
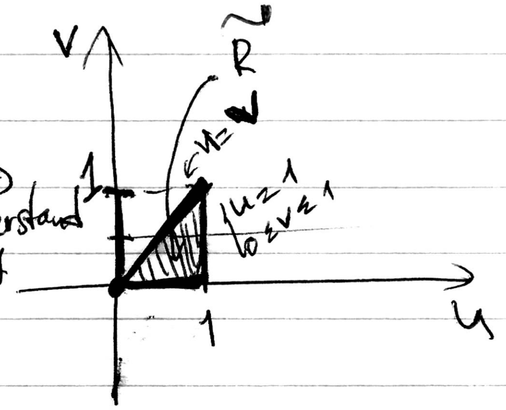
$$f(x,y) = x^2 y = x \cdot xy = u \cdot v = f(x(u,v), y(u,v))$$

3) Find $\tilde{R}$



Draw R

need to understand edges of region



Draw $\tilde{R}$

$$\begin{cases} x = 0 \\ 0 \le y \le 1 \end{cases} \to \begin{cases} u = 0 \\ v = 0 \end{cases}$$

$$\begin{cases} x = 1 \\ 0 \le y \le 1 \end{cases} \to \begin{cases} u = 1 \\ v = y \\ 0 \le y \le 1 \end{cases} \to \begin{cases} u = 1 \\ 0 \le v \le 1 \end{cases}$$

$$\begin{cases} y = 0 \\ 0 \le x \le 1 \end{cases} \to \begin{cases} u = x \\ v = 0 \\ 0 \le x \le 1 \end{cases} \to \begin{cases} 0 \le u \le 1 \\ v = 0 \end{cases}$$

$$\begin{cases} y = 1 \\ 0 \le x \le 1 \end{cases} \to \begin{cases} u = x \\ v = x \\ 0 \le x \le 1 \end{cases} \to \begin{cases} u = v \\ 0 \le u \le 1 \end{cases}$$

4) $$\int_0^1 \int_0^1 x^2 y \, dx\, dy = \int_0^1 \int_v^1 u \cdot v \cdot \frac{1}{u} \, du\, dv = \int_0^1 \int_v^1 v \, du\, dv =$$

$$= \int_0^1 uv \Big|_{u=v}^{u=1} dv = \int_0^1 (v - v^2)\, dv = \left(\frac{v^2}{2} - \frac{v^3}{3}\right)\Big|_0^1 = \boxed{\frac{1}{6}}$$

(9) Example Set up the integral

$\iint_R xy\,dA$, where R is bounded by ellipse $9x^2+4y^2=36$

Using $x=2u$, $y=3v$.

1) $dA = dx\,dy = ?$ in u,v

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 2\cdot 3 - 0\cdot 0 = \boxed{6}$$

$$\underline{dx\,dy} = |6|\,du\,dv = \underline{6\,du\,dv}$$

$$36\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} uv\,du\,dv$$

$$\|$$

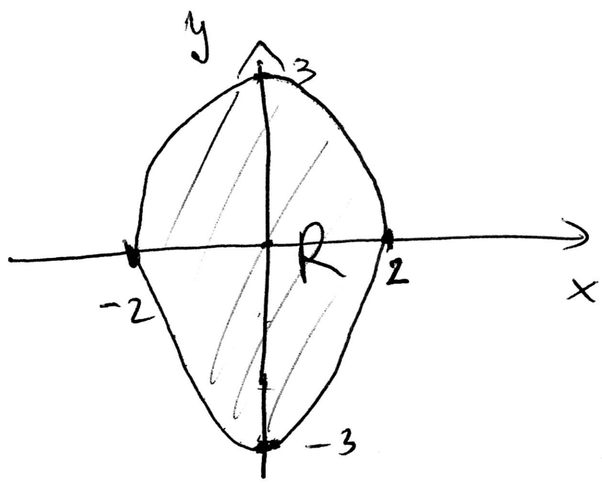
2) $$\iint_R xy\,dA = \iint_{R'} (2u)(3v)\cdot 6\,du\,dv = 36\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} uv\,dv\,du$$

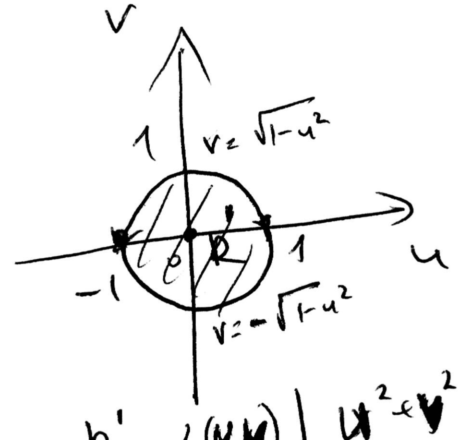
what is R'?

$9x^2+4y^2=36 \to 9\cdot(2u)^2+4(3v)^2=36$

$36u^2+36v^2=36$

$u^2+v^2=1$





$$R' = \{(u,v) \mid u^2+v^2 \le 1\}$$

(10)

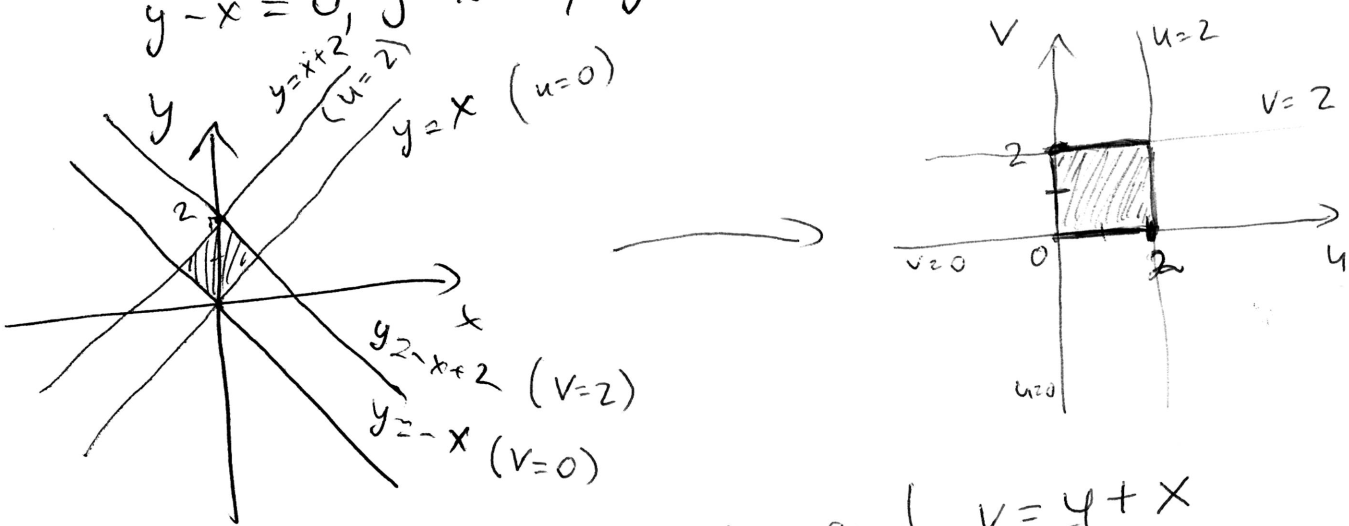
Example

what change of coordinates will simplify the integral? Set up in new coordinates.

$$\iint_R \sqrt{y^2-x^2}\, dA = \iint_R \sqrt{(y-x)(y+x)}\, dA$$

R is the diamond bounded by

$y-x=0$, $y-x=2$, $y+x=0$, $y+x=2$



Solution: 1) Set $u=y-x$ and $v=y+x$

$$dA = dx\,dy = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}\, du\,dv = \frac{1}{2}\, du\,dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -1\cdot 1 - 1\cdot 1 = -2$$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = |-2| = 2$$

2) $$\iint \sqrt{y^2-x^2}\, dA = \iint \sqrt{(y-x)(y+x)}\, dA = \int_0^2\int_0^2 \sqrt{u\cdot v}\cdot \frac{1}{2}\, du\,dv =$$