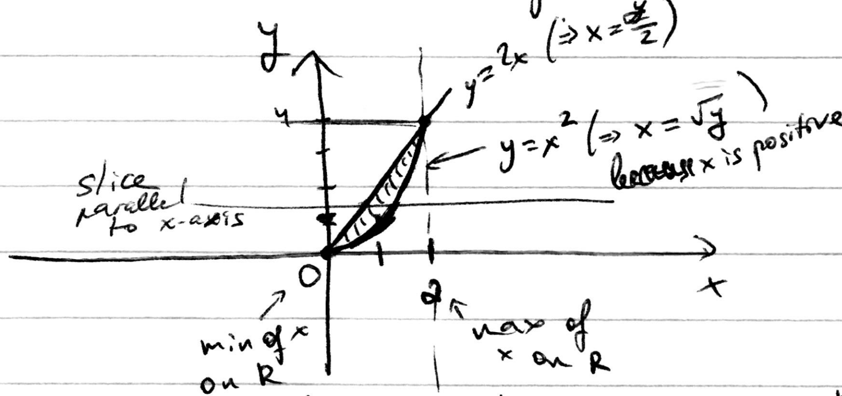
Lecture 6

changing of order of integration

Example

$$\int_0^2 \int_{x^2}^{2x} xy \, dy\, dx = \underbrace{\int \int}_{\text{what are boundaries?}} xy \, dx\, dy$$

Solution: 0) Find out what is R from the integral and Draw R!



1) Write new boundaries using the picture!

$$\iint_R xy\, dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} xy \, dx\, dy$$

~~Example~~

(2)

Example

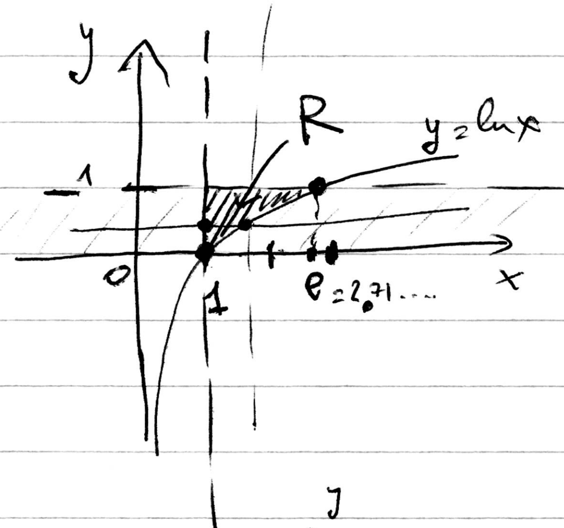
boundaries for x when y fixed

$x = e^y$ max on slice $\to y = \ln x$

$$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy = \underbrace{\iint}_{?} f(x,y)\,dy\,dx$$

$x = 1$ min on slice

Solution: 0) What is R?



tells us that $0 \le y \le 1$

y_{min} on R, y_{max} on R

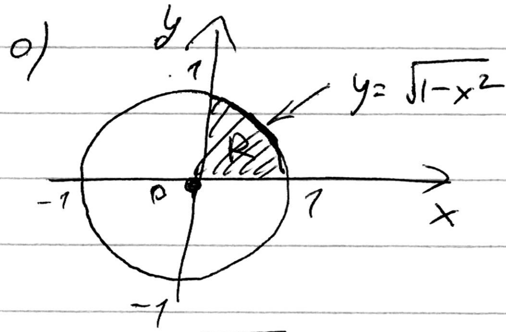
1) $$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy = \int_1^e \int_{\ln(x)}^1 f(x,y)\,dy\,dx$$

③

<u>Example</u>

Set up the integral of $1-x^2-y^2$ over $R = \begin{cases} x \geq 0 \\ y \geq 0 \\ x^2+y^2 \leq 1 \end{cases}$

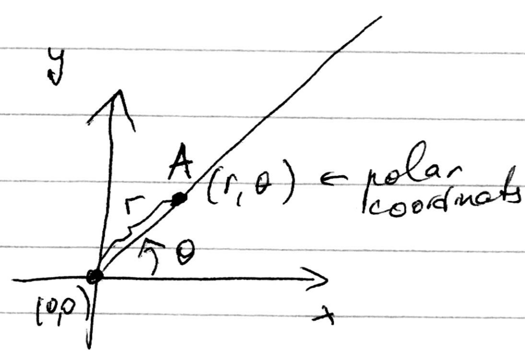
<u>Solution:</u> 0)



1) $$\int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$$

Can we simplify the integral if it is over a part of a disk?

<u>Use polar coordinates!</u> ← useful if either the integrand or the region of integration have a simpler expression



A has polar coordinates (r,θ) if

r = the distance from the point A to $(0,0)$

θ = the angle between the ray connecting $(0,0)$ and the point A and the x-axis (positive − counterclockwise)

Connection between Euclidean (x,y) and polar (r,θ): $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

How to express $\iint_R f(x,y)\,dA$ in polar coordinates?

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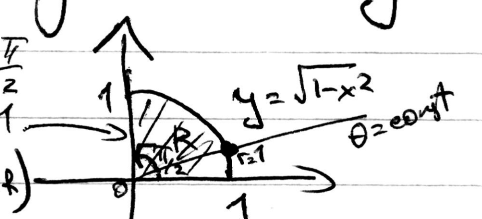
$$dA = \underbrace{dx\,dy = dy\,dx}_{\text{for Euclidean coordinates}} = \underbrace{r\,dr\,d\theta}_{\text{polar coordinates}}$$

$$\iint_R f(x,y)\,dA = \iint_{\substack{R \\ \text{described in polar coordinates}}} f(r\cos\theta, r\sin\theta)\underbrace{r\,dr\,d\theta}_{dA}$$

<u>Example</u>

Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$ using polar coordinates.

$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 1 \end{cases}$



<u>Solution</u>

$$\int_0^1\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx = \int_{\underset{\theta_{min} \text{ on } R}{0}}^{\overset{\theta_{max} \text{ on } R}{\pi/2}} \int_{\underset{r_{min}}{0}}^{\overset{r_{max}}{1}} \left(1-(r\cos\theta)^2-(r\sin\theta)^2\right)\cdot r\,dr\,d\theta$$

(θ max on R (last ray to hit R))

(r_{max} on slice $\theta = \text{const}$.)

θ min on R (first ray to hit R)

boundaries for r and θ now!

r_{min} on slice $\theta = \text{const}$ (ray from $(0,0)$)

$$= \int_0^{\pi/2}\int_0^1 (1-r^2\cos^2\theta - r^2\sin^2\theta)\,r\,dr\,d\theta =$$

$$= \int_0^{\pi/2}\int_0^1 \Big(1-r^2\overbrace{(\cos^2\theta+\sin^2\theta)}^{=1}\Big)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (1-r^2)\,r\,dr\,d\theta =$$

$$= \int_0^{\pi/2}\int_0^1 (r-r^3)\,dr\,d\theta = \int_0^{\pi/2} \left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} = \boxed{\frac{\pi}{8}}$$

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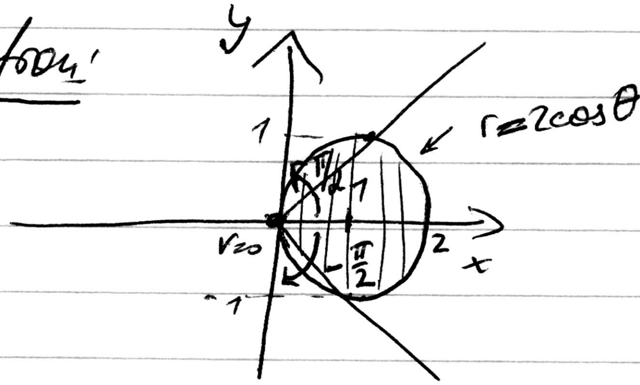
Example

set up $\iint_R \sqrt{x^2+y^2}\, dA$ in polar coordinates,

where R is a region bounded by a circle centered at $(1,0)$ of radius 1.

Solution:



General equation of a circle centered at (a,b) ← Euclidean coordinates

$$(x-a)^2+(y-b)^2=(\text{radius})^2$$

Circle in the problem $(x-1)^2+(y-0)^2=1^2$

$$\Leftrightarrow (x-1)^2+y^2=1$$

R is the region $=\{(x,y) \mid (x-1)^2+y^2\le 1\}$

In polar coordinates

$$(r\cos\theta-1)^2+r^2\sin^2\theta=1$$

$$r^2-2r\cos\theta+1=1$$

$$r^2=2r\cos\theta$$

$$\Downarrow\ r=2\cos\theta$$

or

$$r=0$$

$$\iint_R \sqrt{x^2+y^2}\, dA=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\, r\, dr\, d\theta=$$

$$=\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} r\cdot r\, dr\, d\theta=\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} r^2\, dr\, d\theta$$