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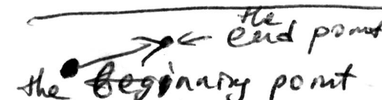
Lecture 11-13

Vector:



Vector field: There is an assigned vector at each point in the plane / space.

In plane (R^2):

Notation: $\bar{i} = \langle 1,0 \rangle$, $\bar{j} = \langle 0,1 \rangle$ "Basis vectors in R^2"

$$\bar{F}(x,y) = P(x,y)\,\bar{i} + Q(x,y)\,\bar{j} = \langle P(x,y), Q(x,y) \rangle$$

vector field

where $P(x,y)$, $Q(x,y)$ – functions.

Examples: 1) velocity vector field

For example, Wind maps show wind flow: $\bar{v} = \bar{v}(x,y)$ wind velocity

– vector depends on a point

2) force vector field:

For example, gravitational field or electric field!

How to draw vector field:

Example: Draw $\bar{F} = 2\bar{i} + \bar{j}$ ← at each point have vector $\langle 2,1 \rangle$

Pick sample of points and draw vectors of F at those points, i.e., compute $\bar{F}(x_0,y_0)$ at a point (x_0,y_0) and then draw vector $\bar{F}(x_0,y_0)$ with beginning at (x_0,y_0)

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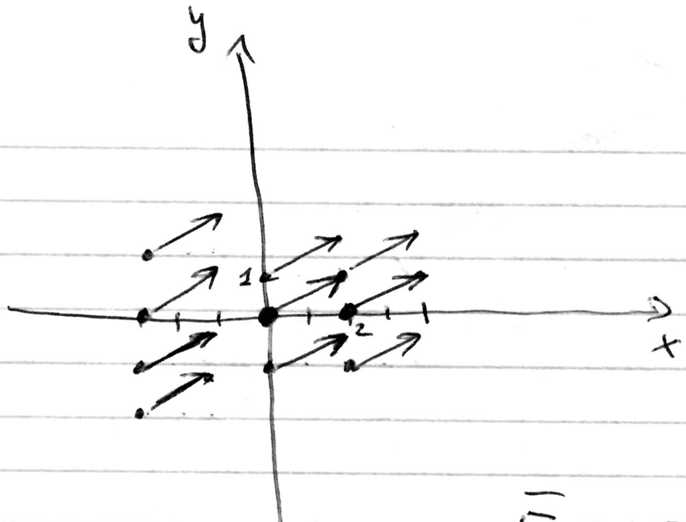
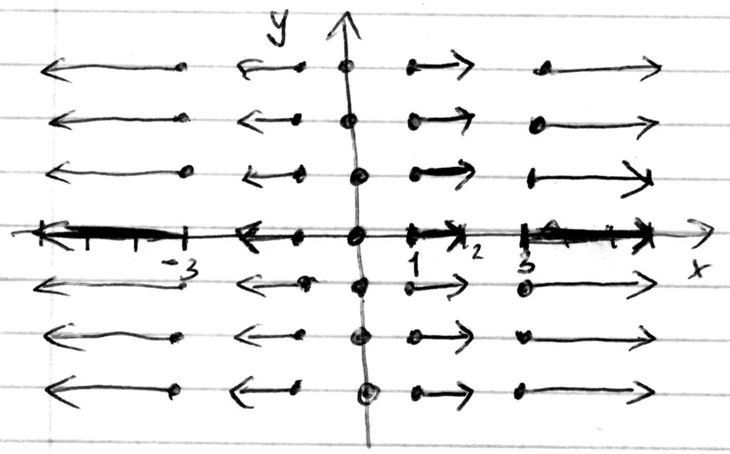


<u>Example</u> Draw $\bar{F} = x\,\bar{i} \;\left(= \langle x, 0 \rangle \right)$



$(0,0) \longrightarrow \bar{F}(0,0) = \langle 0, 0 \rangle$

$(0,1) \longrightarrow \bar{F}(0,1) = \langle 0, 0 \rangle$

For any a $\quad (0,a) \longrightarrow \bar{F}(0,a) = \langle 0, 0 \rangle$

$(1,0) \rightarrow \bar{F}(1,0) = \langle 1, 0 \rangle$

Fo any a $\quad (1,a) \rightarrow \bar{F}(1,a) = \langle 1, 0 \rangle$

$(-1,a) \longrightarrow \bar{F}(-1,a) = \langle -1, 0 \rangle$

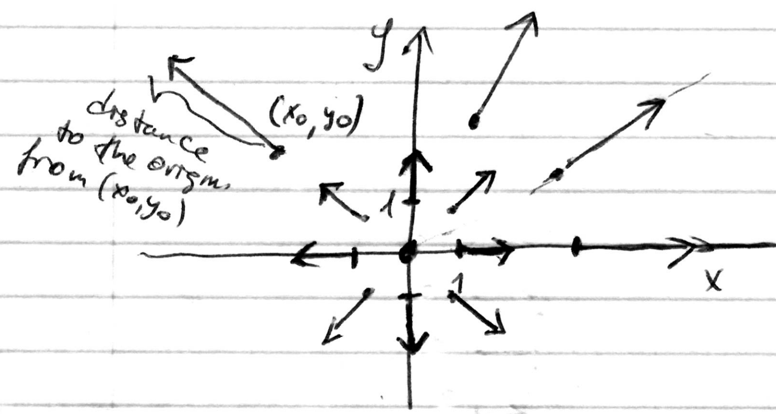
$(3,a) \rightarrow \bar{F}(3,a) = \langle 3, 0 \rangle$

$(-3,a) \rightarrow \bar{F}(-3,a) = \langle -3, 0 \rangle$

(3)

Length of a vector $\bar{v} = a\bar{i} + b\bar{j}$ $(= \langle a, b\rangle)$
is equal to $\underbrace{|\bar{v}|}_{\text{notation}} = \sqrt{a^2+b^2}$

Example Sketch $\bar{F} = x\bar{i} + y\bar{j}$



$(0,0) \to \bar{F}(0,0) = \langle 0, 0\rangle$
$(1,0) \to \bar{F}(1,0) = \langle 1, 0\rangle$
$(-1,0) \to \bar{F}(-1,0) = \langle -1, 0\rangle$
$(0,1) \to \bar{F}(0,1) = \langle 0, 1\rangle$
$(0,-1) \to \bar{F}(0,-1) = \langle 0, -1\rangle$

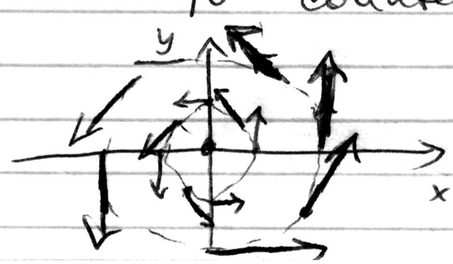
magnitude of F $\to$ $|F| = \sqrt{x^2+y^2}$ $\leftarrow$ distance to the origin from point (x,y).

(Further point from the origin, bigger the magnitude)

direction of $\bar{F}$ = radially outwards

Example: $\bar{F} = -y\bar{i} + x\bar{j}$

$\langle -y, x\rangle$ is the vector $\langle x, y\rangle$ rotated by 90° counterclockwise



(4) <u>In space</u> $(\mathbb{R}^3)$: vector $\bar{v} = \langle a, b, c\rangle$
length of $\bar{v} = |\bar{v}| = \sqrt{a^2+b^2+c^2}$

<u>Notation:</u> $\bar{i} = \langle 1, 0, 0\rangle$
$\bar{j} = \langle 0, 1, 0\rangle$
$\bar{k} = \langle 0, 0, 1\rangle$

$$\bar{F}(x,y,z) = P(x,y,z)\bar{i} + Q(x,y,z)\bar{j} + R(x,y,z)\bar{k}$$

vector field where P, Q, R – functions of x, y, z.

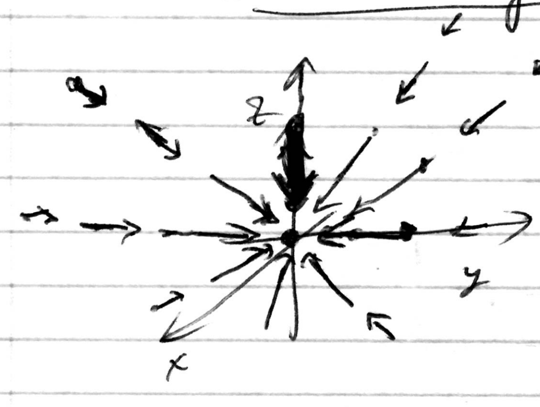
<u>Example</u> Gravitational field of a mass at $(0,0,0)$

not defined at $(0,0,0)$. $\rightarrow$ $\bar{F} = \left\langle -\frac{cx}{(x^2+y^2+z^2)^{3/2}}, \frac{-cy}{(x^2+y^2+z^2)^{3/2}}, -\frac{cz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$

<u>where $c>0$</u>

$$|\bar{F}| = \frac{c}{x^2+y^2+z^2} = \frac{c}{\rho^2}$$, where ρ – the distance to the origin

<u>Direction of $\bar{F}$</u>: towards the origin $(0,0,0)$ radial



Bigger ρ, Smaller $|\bar{F}|$

Example (Gradient vector field)

Let $f(x,y)$ be a function of x and y

Then $\overline{F} = \nabla f = \langle f_x, f_y \rangle$ - gradient vector field

gradient

f is called potential function for $\overline{F}$

!!! Not every vector field is gradient vector field

Definition: $\overline{F}$ is said to be conservative vector field if there exists function $f(x,y)$ s.t. $\overline{F} = \nabla f$.

(6)

Line integrals

How to integrate a quantity over a $\underset{\substack{\uparrow \\ \text{1-dimensional}}}{\underline{\text{curve}}}$?

Setting 1 Given a function $f(x,y)$ and C - a curve in the plane

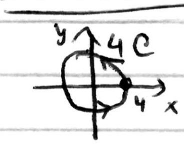
Solution 1. Parametrize the curve C, i.e. describe C as
$\bar{r}(\underset{\substack{\uparrow \\ \text{parameter}}}{t}) = \langle x(t), y(t) \rangle$ for $a \le t \le b$

2. $$\underbrace{\int_C f(x,y)\, \overset{\text{arclength}}{\overline{ds}}}_{\text{line integral}} = \underbrace{\int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{\text{standard integral}}$$

parametrization for C

Example: Compute
$$\int_C (x^2+y^2)\, ds$$ where C is the circle of radius 4 centered at $(0,0)$

Solution: 1. Parametrize C:



$$\bar{r}(t) = \langle \overbrace{4\cos(t)}^{x(t)}, \overbrace{4\sin(t)}^{y(t)} \rangle, \quad 0 \le t \le 2\pi$$

(Recall in polar coordinates C is $r=4$
Connection between (x,y) and (r,θ) is
$x = r\cos\theta$
$y = r\sin\theta$)

$$\frac{dx}{dt} = -4\sin(t)$$

$$\frac{dy}{dt} = 4\cos(t)$$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} \left((4\cos(t))^2 + (4\sin(t))^2 \right) \sqrt{(-4\sin(t))^2 + (4\cos(t))^2}\,dt$$

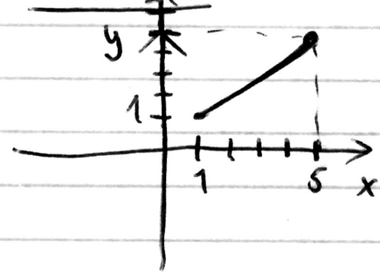
$$= \int_0^{2\pi} 16\,\underbrace{(\cos^2(t)+\sin^2(t))}_{=1} \sqrt{16\underbrace{(\sin^2(t)+\cos^2(t))}_{=1}}\,dt =$$

$$= \int_0^{2\pi} 16\cdot 4\,dt = \int_0^{2\pi} 64\,dt = 64t\Big|_0^{2\pi} = 128\pi$$

<u>Example</u> Compute $\int_C (x^2+y^2)\,ds$ where

C is the line segment from [redacted] (1,1) to (5,5).

<u>Solution:</u> 1. Parametrize C:



Parametric equation of a line from (a,b) to (c,d)

$$\begin{cases} x(t) = a + (c-a)t \\ y(t) = b + (d-b)t \end{cases}$$

where $0 \le t \le 1$

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Our C: $\vec{r}(t) = \langle x(t), y(t) \rangle$, where

$$\begin{cases} x(t) = 1 + (5-1)t \\ y(t) = 1 + (5-1)t \end{cases} \rightarrow \begin{cases} x(t) = 1+4t \\ y(t) = 1+4t \end{cases}$$

$$\frac{dx}{dt} = 4, \quad \frac{dy}{dt} = 4$$

2.

$$\int_C (x^2+y^2)\,ds = \int_0^1 \left((1+4t)^2 + (1+4t)^2\right)\sqrt{4^2+4^2}\,dt =$$

$$= \int_0^1 2(1+4t)^2 \cdot 4\sqrt{2}\,dt = 2\sqrt{2}\int_1^5 u^2\,du =$$

u-substitution
$u = 1+4t$
$du = 4dt$
$t=0 \rightarrow u=1$
$t=1 \rightarrow u=5$

$$= 2\sqrt{2}\,\frac{u^3}{3}\Big|_1^5 = 2\sqrt{5}\left(\frac{5^3}{3} - \frac{1}{3}\right) = \frac{248\sqrt{5}}{3}$$

Can you suggest other parametrizations of C? Yes,
$$\begin{cases} x(t) = t \\ y(t) = t \end{cases} \quad \text{where } 1 \le t \le 5$$

Then $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$

$$\int_C (x^2+y^2)\,ds = \int_1^5 (t^2+t^2)\sqrt{1^2+1^2}\,dt =$$

$$= 2\sqrt{2}\int_1^5 t^2\,dt = 2\sqrt{2}\,\frac{t^3}{3}\Big|_1^5 = \frac{248\sqrt{5}}{3}$$

⑨

! Result doesn't depend on the parametrization of a curve C

! There are a lot of parametrizations of the same curve.

! Calculations might be simpler, depending on the parametrization,

Similar in $\mathbb{R}^3$

Given $f(x,y,z)$ - function
C - curve in $\mathbb{R}^3$ with parametrization
$\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$
where $a \le t \le b$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Property: If C consists of pieces $C_1, C_2, \dots, C_3$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_k} f\,ds$$

Work and line integrals

Setting 2: Compute the work W done by a force $\bar{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ on an object moving along a curve C.

Solution: 1. Let C be parametrized as $\bar{r}(t) = \langle x(t), y(t)\rangle$ where $a \le t \le b$.

2. $$\underset{\text{work}}{W} = \int_C \bar{F} \cdot d\bar{r} \underset{\text{compute}}{=} \int_a^b \left(\bar{F}(x(t), y(t)) \cdot \frac{d\bar{r}}{dt}\right) dt$$

where $\frac{d\bar{r}}{dt} = \langle x'(t), y'(t)\rangle$
derivatives

$$\bar{F}(x(t), y(t)) \cdot \frac{d\bar{r}}{dt} = P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)$$

Example: Compute work done by $\bar{F} = -y\bar{i} + x\bar{j}$ along $C: \bar{r}(t) = \langle t, t^2\rangle$, where $0 \le t \le 1$

Solution: C is $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$ $0 \le t \le 1$

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_0^1 \left(\bar{F}(t, t^2) \cdot \frac{d\bar{r}}{dt}\right) dt = \int_0^1 \left(-t^2 \cdot 1 + t \cdot 2t\right) dt$$

because

$\bar{F}(t, t^2) = \langle -t^2, t\rangle$ as $P(x,y) = -y$, $Q(x,y) = x$

$$\frac{d\bar{r}}{dt} = \langle x'(t), y'(t)\rangle = \langle 1, 2t\rangle$$

$$\bar{F}(t, t^2) \cdot \frac{d\bar{r}}{dt} = -t^2 \cdot 1 + t \cdot 2t$$

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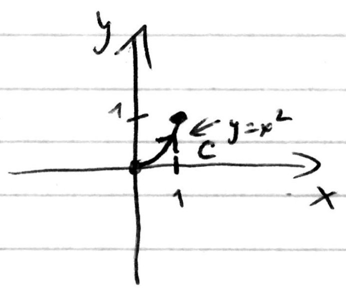
$$W = \int_0^1 (-t^2+2t^2)\,dt = \int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$$

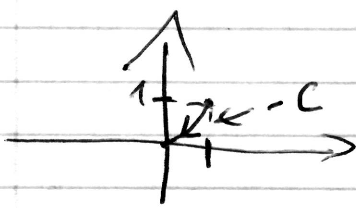
! Orientation of curve matters!

$-C$ is C in opposite direction



$\tilde{t} = 1-t$ when ~~t~~ t varies from 0 to 1, then $\tilde{t}$ from 1 to 0.

$-C$ can be parametrized as



$$\begin{cases} x(t) = 1-t \\ y(t) = (1-t)^2 \end{cases} \quad 0 \le t \le 1$$

$$\bar{F}(x(t), y(t)) = \langle -(1-t)^2,\ 1-t \rangle$$

$$\frac{d\bar{r}}{dt} = \langle -1,\ -2(1-t) \rangle$$

$$W = \int_{-C} \bar{F}\cdot d\bar{r} = \int_0^1 -(1-t)^2\cdot(-1) + (1-t)(-2(1-t))\,dt =$$

$$= -\int_0^1 (1-t)^2\,dt \underset{\uparrow}{=} \int_1^0 u^2\,du = \frac{u^3}{3}\Big|_1^0 = 0 - \frac{1}{3} = \boxed{-\frac{1}{3}}$$

$u = 1-t$

$du = -dt$

$t=0 \to u=1,\ t=1 \to u=0$

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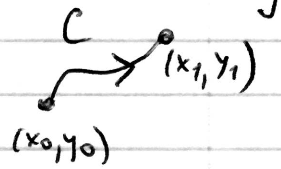
Fundamental theorem

Thm Let $\bar{F} = \nabla f$ ← conservative vector field



If C - a curve that runs from (x_0, y_0) to (x_1, y_1), then

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r} = f(x_1, y_1) - f(x_0, y_0)$$

Example 1. Calculate work of $\bar{F} = x\bar{i} + y\bar{j}$ along C: $\bar{r}(t) = \begin{cases} x(t) = t \\ y(t) = 0 \end{cases}$ and $0 \le t \le 1$

Solution 1:

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_0^1 \langle t, 0 \rangle \cdot \langle 1, 0 \rangle \, dt =$$

$$= \int_0^1 (t \cdot 1 + 0 \cdot 0) \, dt = \int_0^1 t \, dt = \frac{t^2}{2} \Big|_0^1 = \boxed{\frac{1}{2}}$$

Solution 2: Notice that $\bar{F} = x\bar{i} + y\bar{j}$ is equal to $\bar{F} = \nabla f$ where $f = \frac{1}{2}(x^2 + y^2)$

Recall $\nabla f = \langle f_x, f_y \rangle = \langle \frac{2x}{2}, \frac{2y}{2} \rangle = \langle x, y \rangle$

$$W = \int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r} = f(\text{end of } C) - f(\text{beginning of } C)$$

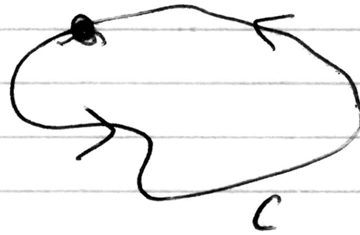
C runs from $\underbrace{(0,0)}_{\text{beginning}}$ to $\underbrace{(1,0)}_{\text{end}}$

plug $t=0$ into parametrization of C

plug $t=1$ into parametrization of C

$$W = f(1,0) - f(0,0) = \frac{1}{2}(1^2+0^2) - \frac{1}{2}(0^2+0^2) = \boxed{\frac{1}{2}}$$

What if $\bar{F} = \nabla f$ and C is a closed curve, i.e. the beginning of C = the end of C ?



$$\begin{array}{l}\text{Work}\\ \text{along}\\ \text{closed}\\ \text{curve}\end{array} = \underset{C}{\overset{\text{closed}}{\oint}} \nabla f \cdot d\bar{r} = f(\text{end}) - f(\text{begin}) =$$

$$= \boxed{0} \text{ as end = begin}$$

$$\bar{F} = \nabla f,\ C\text{-closed} \Rightarrow \int_C \bar{F} \cdot d\bar{r} = 0$$

How to find out if $\bar{F}$ is conservative?

Theorem Let $\bar{F} = \langle P(x,y), Q(x,y) \rangle$ be continuously differentiable ("nice") in $\mathbb{R}^2$.

Then $\bar{F}$ is conservative if and only if

$$P_y = Q_x \text{ at every point.}$$

Why $P_y = Q_x$ has to hold?

If $\bar{F} = \nabla f = \langle \overset{\text{"}P\text{"}}{f_x}, \overset{\text{"}Q\text{"}}{f_y} \rangle \leftarrow$ "nice"

Helps to remember $\to$ $$P_y = (f_x)_y = f_{xy} \underset{\uparrow\ \text{"nice"}}{=} f_{yx} = (f_y)_x = Q_x$$

Example Is $\bar{F} = -y\bar{i} + x\bar{j}$ conservative?

Solution: $P(x,y) = -y \Rightarrow P_y = -1$
$Q(x,y) = x \qquad Q_x = 1$ $\Big) \Rightarrow$ not conservative

Example Is $\bar{F} = (3x^2 + 8xy)\bar{i} + (4x^2 + 3y^2)\bar{j}$ conservative?

Solution: $P(x,y) = 3x^2 + 8xy \Rightarrow P_y = 8x$
$Q(x,y) = 4x^2 + 3y^2 \qquad Q_x = 8x$ $\Big) \Rightarrow$ Conservative

How to find potential if $\bar{F}$ is conservative?

$$\begin{cases} \bar{F} = \langle P(x,y), Q(x,y) \rangle \\ \bar{F} \text{ - conservative, i.e. } \bar{F} = \langle f_x, f_y \rangle \end{cases}$$

Then $\begin{cases} f_x = P(x,y) \\ f_y = Q(x,y) \end{cases} \xrightarrow{\text{solve}}$ get f.

Example We saw $\bar{F} = (3x^2 + 8xy)\bar{i} + (4x^2 + 3y^2)\bar{j}$ is conservative. Find potential:

Solution: Need to solve

$$\begin{cases} f_x = 3x^2 + 8xy & (1) \\ f_y = 4x^2 + 3y^2 & (2) \end{cases}$$

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How to solve?

1) Use (1) equation and integrate with respect to x:

$$f_x = 3x^2 + 8xy$$

$$\Downarrow \quad f(x,y) = \int (3x^2 + 8xy)\,dx = x^3 + 4x^2 y + \underbrace{C(y)}_{\text{function of } y \text{ is constant for } x}$$

indefinite

2) Plug in expression for f from step 1 into 2 and find C(y) by integration with respect to y

$$4x^2 + 3y^2 = f_y = 4x^2 + C'(y) \implies 4x^2 + 3y^2 = 4x^2 + C'(y)$$

$$C'(y) = 3y^2$$ depends only on y!

Integrate w.r.t. y

$$C(y) = \int 3y^2\,dy = y^3 + \tilde{C}$$ true constant

3) Combine results of step 1 and step 2 to find f

$$f(x,y) = x^3 + 4x^2 y + y^3 + \tilde{C}$$ — any such function is a potential for

$$\overline{F} = \langle 3x^2 + 8xy,\ 4x^2 + 3y^2 \rangle$$

Flux and line integrals

Setting 3

Compute flux of a vector field $\overline{F}$ across curve C.

$$\text{Flux} = \int_C (\overline{F} \cdot \overline{n})\, ds,$$

where $\overline{n}$ = the unit normal vector to C
(the outward if C is closed)

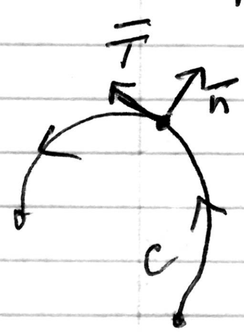
and $\overline{F} \cdot \overline{n}$ = dot product of $\overline{F}$ and $\overline{n} = P\alpha + Q\beta$

if $\overline{F} = \langle P, Q \rangle$
$\overline{n} = \langle \alpha, \beta \rangle$

Interpretation:

If $\overline{F}$ is a velocity field in wind flow then flux measure how much matter (wind) passes through C per unit time.

How to find $\overline{n}$ if C is parametrized



$$\overline{F}(t) = \langle x(t), y(t) \rangle ? \quad a \le t \le b$$

$$\overline{n} = \frac{\langle y'(t), -x'(t) \rangle}{\sqrt{(y'(t))^2 + (x'(t))^2}} = \left\langle \frac{y'(t)}{\sqrt{(x')^2 + (y')^2}}, \frac{-x'(t)}{\sqrt{(x')^2 + (y')^2}} \right\rangle$$

Notice $\overline{n}$ is orthogonal to the unit tangent vector

$$\overline{T} = \left\langle \frac{x'(t),\ y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \right\rangle$$

to C because

$$\overline{n} \cdot \overline{T} = 0$$

dot product.

Recall $ds = \sqrt{(x'(t))^2 + (y'(t))^2}$

$$\bar{n} = \frac{\langle y'(t), -x'(t)\rangle}{\sqrt{(x'(t))^2+(y'(t))^2}}$$

Therefore:

$$\text{Flux} = \int_C (\bar{F}\cdot\bar{n})\,ds = \int_a^b \Big(P(x(t),y(t))\cdot y'(t) - Q(x(t),y(t))\cdot x'(t)\Big)\,dt$$

where $\bar{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$

C is $\bar{r}(t) = \langle x(t), y(t)\rangle$ and $a \le t \le b$

Example Let $F = \langle \overset{=P}{y}, \overset{=Q}{x}\rangle$ and

$C: r(t) = \langle \overset{=x(t)}{2\cos(t)}, \overset{=y(t)}{2\sin(t)}\rangle$, $0 \le t \le 2\pi$.

compute $\bar{n}$. Compute flux of $\bar{F}$ through C.

Solution

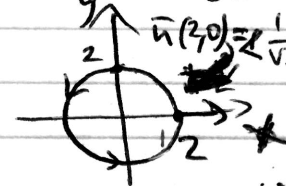
$\text{Flux} = \int_C (\bar{F}\cdot\underset{\text{outward}}{\bar{n}})\,ds$

1) $x'(t) = -2\sin(t)$

$y'(t) = 2\cos(t)$

$$\bar{n} = \frac{\langle 2\cos(t), -(-2\sin t)\rangle}{\sqrt{(-2\sin(t))^2+(2\cos(t))^2}} = \frac{\langle 2\cos(t), 2\sin t\rangle}{2\sqrt{2}} = \frac{\langle \cos(t), \sin(t)\rangle}{\sqrt{2}}$$

(Note $(-\bar{n})$ – the inward unit vector)



$$\text{Flux} = \int_C (\bar{F}\cdot\bar{n})\,ds = \int_0^{2\pi}\Big(2\sin(t)\cdot 2\cos(t) - (2\cos(t))\cdot(-2\sin(t))\Big)\,dt =$$

$$= \int_0^{2\pi} 8\sin(t)\cos(t)\,dt = 4\int_0^{2\pi}\sin(2t)\,dt = -2\cos(2t)\Big|_0^{2\pi} =$$

$$= -2-(-2) = 0$$