Lecture 3

Absolute (Global) max & min values

Def

Abs. max at (a,b) $\longrightarrow$ $f(x,y) \le f(a,b)$ for every (x,y) in the domain of f

Abs. min at (a,b) $\longrightarrow$ $f(x,y) \ge f(a,b)$ for every (x,y) in the domain of f.

Example Find the absolute max & min values of f on R, where $f(x,y) = x^2+y^2-2y+1$ and $R = \{(x,y) \mid x^2+y^2 \le 4\}$

Solution: 1. Find critical points:

$$\begin{cases} f_x = 2x = 0 \\ f_y = 2y-2 = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases}$$

$\rightarrow (0,1)$ – the only critical point

2. Classify critical point (Second derivative test?)

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$

$D(x,y) = 2 \cdot 2 - 0^2 = 4$

$$\left.\begin{array}{l} D(0,1) = 4 > 0 \\ f_{xx}(0,1) = 2 > 0 \end{array}\right\} \rightarrow (0,1) \text{ – \underline{local} min}$$

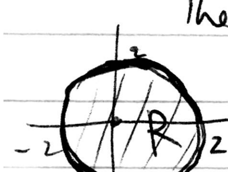
$\rightarrow f(0,1) = 0^2+1^2-2 \cdot 1+1 = 0$ – local min value

3. Where is max?

Must be along boundary!

Is local min is abs. min? Check the boundary!

The boundary of R is $\{(x,y) \mid x^2+y^2 = 4\}$



$\rightarrow$ "plug in" the boundary condition into f after solving the constraint,

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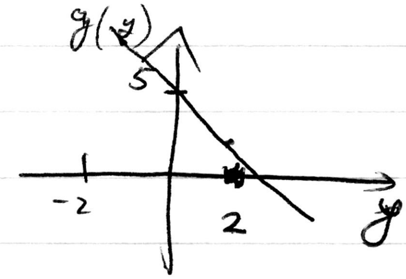
$$\begin{cases} f(x,y) = x^2 + y^2 - 2y + 1 \\ x^2 + y^2 = 4 \longrightarrow x^2 = 4 - y^2 \to x = \pm\sqrt{4-y^2} \end{cases}$$

On the boundary we analyze the function

$g(y) = 4 - y^2 + y^2 - 2y + 1 = 5 - 2y$

$g'(y) = -2 < 0 \to g(y)$ – decreasing function

From R we see that $-2 \le y \le 2$



$g(-2) = 5 - 2\cdot(-2) = 5 + 4 = 9$

$g(2) = 5 - 2\cdot 2 = 5 - 4 = 1$

On boundary $x = \pm\sqrt{4-y^2} \to$ if $y = -2$, then $x = \pm 0 = 0$

Max on the boundary is 9

Min on the boundary is $1 > f(0,1) = 0$

<u>Answer:</u> Abs. max is 0 at (0, 1)

Abs. min is 9 at (0, -2)

<u>Finding Abs Max/Min Values on Closed Bounded Sets</u>

Let f – continuous on R

1. Find critical points in R (& their type) not necessary but good for checking work
2. Determine values of f at all critical points in R.
3. Find the max & min values of f on the boundary of R — Not always as easy as in the example.
4. The greatest function value found in 1-3 – abs. max value

The least function value found in 1-3 – abs min value

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Lagrange Multipliers

Settings

0) $f(x,y) \to$ min/max value with constraints $g(x,y)=0$ ← we want

1) f, g – differentiable on a region of $\mathbb{R}^2$

3) $\nabla g(x,y) \neq \langle 0,0 \rangle$ on the curve $g(x,y)=0$.

Steps to solve: 1. Find the values of x, y, λ (if exist) that satisfy the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y)=0$$

(Remark: λ – some constant you want to find) — Lagrange multiplier

2. Among the values at (x,y) found in (1) select the largest → max value satisfying $g(x,y)=0$

the smallest → min value satisfying $g(x,y)=0$

Lecture 4

Example Find min/max value of $f(x,y)=x^2+y^2-2y+1$ satisfying $x^2+y^2=4$.

Solution: $g(x,y) = x^2+y^2-4$ (Constraint $g(x,y)=0$)

$$\nabla f = \langle 2x, 2y-2 \rangle$$
$$\nabla g = \langle 2x, 2y \rangle$$

Equation $\nabla f = \lambda \nabla g$ means = $\begin{cases} 2x = \lambda \cdot 2x \\ 2y-2 = \lambda \cdot 2y \end{cases}$

$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$

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Solve system $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$:

$$\begin{cases} 2x = \lambda \cdot 2x \\ 2y - 2 = \lambda \cdot 2y \\ x^2 + y^2 - 4 = 0 \end{cases} \rightarrow \begin{cases} 2x(1-\lambda) = 0 \\ 2y(1-\lambda) - 2 = 0 \\ x^2 + y^2 = 4 \end{cases}$$

$$2x(1-\lambda) = 0 \quad \text{or} \quad \begin{cases} \nearrow x = 0 \longrightarrow \begin{cases} x = 0 \\ x^2 + y^2 = 4 \\ 2y(1-\lambda) - 2 = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = \pm 2 \\ 2y(1-\lambda) = 2 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 2 \\ \lambda = 1/2 \end{cases} \text{ or } \begin{cases} x = 0 \\ y = -2 \\ \lambda = \frac{3}{2} \end{cases} \\ \searrow \lambda = 1 \rightarrow \begin{cases} \lambda = 1 \\ 0 - 2 = 0 \leftarrow \text{not possible} \rightarrow \text{no solution.} \\ x^2 + y^2 = 4 \end{cases} \end{cases}$$

$f(0,2) = 0^2 + 2^2 - 2 \cdot 2 + 1 = 0 \rightarrow$ min on $g(x,y) = 0$

$f(0,-2) = 0^2 + (-2)^2 - 2 \cdot (-2) + 1 = 9 \rightarrow$ max on $g(x,y) = 0$

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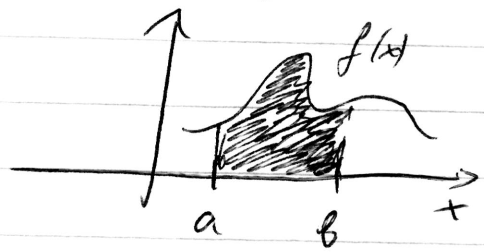
Double integrals

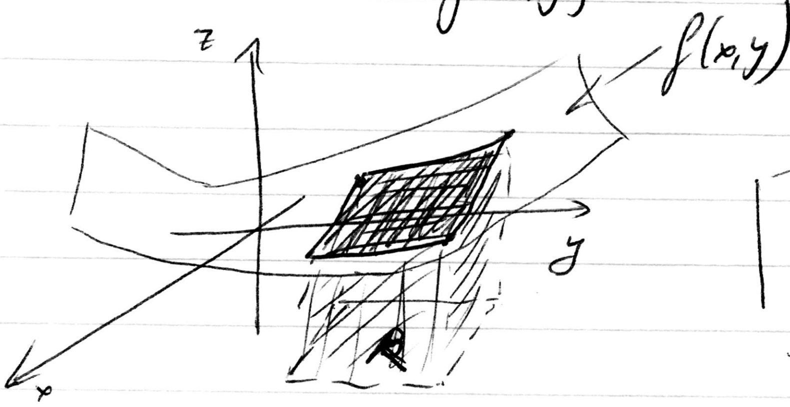
$f(x) \geq 0$
what is $\int_a^b f(x)dx$? It is the area under $f(x)$,

i.e., the area of [shaded] on the picture



$f(x,y) \geq 0$
what is $\iint_R f(x,y)\,dA$, wher R – region in the xy-plane?

It is the volume under $f(x,y)$.



$$\boxed{\text{Area of } R = \iint_R 1\,dA}$$

If $f(x,y)$ not necessarily ≥ 0, then we have "signed area".

(Fubini's theorem)

f – continuous

$R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$ "notation" $= [a,b] \times [c,d]$

then,

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

iterated integrals.

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What means $\int_1^2 (2x+y)\,dx$?

Treat y as a constant, integrate with respect to x.

$$\int_1^2 (2x+y)\,dx = (x^2+yx)\Big|_{x=1}^{x=2} = 2^2 + y\cdot 2 - (1^2 + y\cdot 1) =$$
$$= 4 + 2y - 1 - y = \boxed{3+y}$$

function of y

what means $\int_1^2 (2x+y)\,dy$?

Treat x as a constant, integrate with respect to y.

$$\int_1^2 (2x+y)\,dy = \left(2xy + \frac{y^2}{2}\right)\Big|_{y=1}^{y=2} = 2x\cdot 2 + \frac{2^2}{2} - \left(2x\cdot 1 + \frac{1}{2}\right) =$$
$$= 4x + 2 - 2x - \frac{1}{2} =$$
$$= \boxed{2x + \frac{3}{2}}$$

function of x.

Example ~~Evaluate~~

Evaluate $\iint_R 6xy^2\,dA$, where $R = \underset{x}{[2,4]} \times \underset{y}{[1,2]}$

Solution:

$$\iint_R 6xy^2\,dA = \int_1^2 \int_2^4 6xy^2\,dx\,dy = \int_1^2 \left[\int_2^4 6xy^2\,dx\right] dy$$

1) $$\int_2^4 6xy^2\,dx = 3x^2y^2\Big|_{x=2}^{x=4} = 3\cdot 4^2 y^2 - 3\cdot 2^2\cdot y^2 =$$
$$= 36y^2$$

2) $$\int_1^2 36y^2\,dy = 12y^3\Big|_{y=1}^{y=2} = 12\cdot 2^3 - 12\cdot 1^3 = 12\cdot 7 = \boxed{84}$$

$\iint_R 6xy^2\,dA = 84$

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Could compute $\iint_R 6xy^2\,dA = \int_2^4\left[\int_1^2 6xy^2\,dy\right]dx$

Lecture 5

Question: what is $\iint_R 1\,dA$?

It is equal to the area of R!

Example

Evaluate $\iint_R \cancel{x}\, e^{xy}\,dA$, where $R = [-1,2]\times[0,1]$

Solution:

$\iint_R xe^{xy}\,dA = \int_0^1\int_{-1}^2 xe^{xy}\,dx\,dy \quad \rightsquigarrow \quad 1)\ \int_{-1}^2 xe^{xy}\,dx \to$ hard!

$\iint_R xe^{xy}\,dA = \int_{-1}^2\left[\int_0^1 xe^{xy}\,dy\right]dx$

1) $\int_0^1 xe^{xy}\,dy = e^{xy}\Big|_{y=0}^{y=1} = e^x - 1$

2) $\int_{-1}^2 (e^x-1)\,dx = (e^x - x)\Big|_{x=-1}^{x=2} = e^2 - 2 - (e^{-1} - (-1)) =$
$= e^2 - e^{-1} - 3$

$\iint_R xe^{xy}\,dA = \boxed{e^2 - e^{-1} - 3}$

Remark: sometimes the order is important!

Average Value of f over region R

$$\bar{f} = \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA$$

Example Average value of $f = 6xy^2$ on $R = [2,4]\times[1,2]$

area of $R = 2\cdot 1 = 2$

$\bar{f} = \frac{1}{\text{area of } R}\iint_R 6xy^2\,dA = \frac{1}{2}\cdot 84 = \boxed{42}$

8 **Properties**

$$\iint_R f(x,y)+g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

R - splits into two regions R_1 and R_2

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

How to compute if R is not rectangle?

$$\int_{\substack{\text{min of } y\\ \text{on } R}}^{\substack{\text{max of } y\\ \text{on } R}} \left[\int_{\substack{\text{min of } x\\ \text{on slice}}}^{\substack{\text{max of } x\\ \text{on slice}}} f(x,y)\,dx\right] dy$$

boundaries for x while y is some constant (what does slice parallel to x-axis look like?)

- function of y

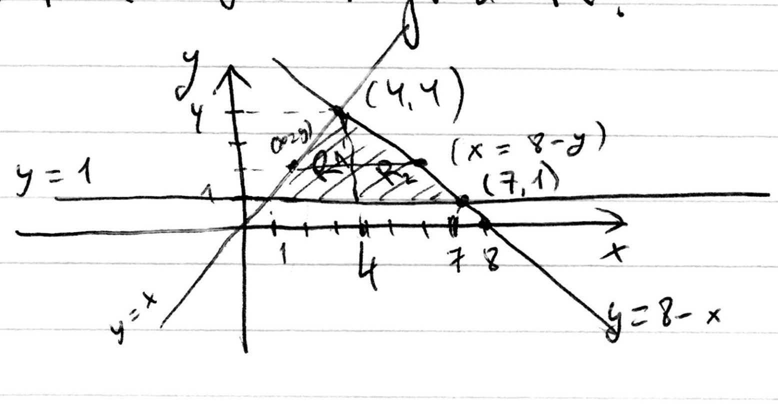
$$\int_{\substack{\text{min of } x\\ \text{on } R}}^{\substack{\text{max of } x\\ \text{on } R}} \left[\int_{\substack{\text{min of } y\\ \text{on slice}}}^{\substack{\text{max of } y\\ \text{on slice}}} f(x,y)\,dy\right] dx$$

boundaries for y while x is some constant (what does slice parallel to y-axis look like?)

- function of x.

Example Compute $\iint_R \left(12+\frac{1}{y}\right) dA$, where R - region in xy-plane bounded by $y=x$, $y=8-x$ and $y=1$.

Solution: 0) Draw the region R!



Intersection of $y=x$ & $y=8-x$

$$\begin{cases} y=x \\ y=8-x \end{cases} \to \begin{cases} y=x \\ x=8-x \end{cases} \to \begin{cases} y=4 \\ x=4 \end{cases}$$

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One way:

$$\int_1^4\int_y^{8-y}\left(2+\frac{1}{y}\right)dx\,dy = \int_1^4\left(2x+\frac{x}{y}\right)\Bigg|_{x=y}^{x=8-y} dy =$$

$$= \int_1^4\left(\left(2(8-y)+\frac{8-y}{y}\right)-\left(2y+\frac{y}{y}\right)\right)dy =$$

$$= \int_1^4\left(16-2y+\frac{8}{y}-1-2y-1\right)dy =$$

$$= \int_1^4\left(14-4y+\frac{8}{y}\right)dy = 14y-2y^2+8\ln|y|\,\Bigg|_{y=1}^{y=4} =$$

$$= 14\cdot4-2\cdot4^2+8\ln4-\left(14\cdot1-2\cdot1^2+8\cdot\underset{\overset{\parallel}{0}}{\underline{\ln 1}}\right) =$$

$$= 56-32+8\ln4-14+2-0 = \boxed{12+8\ln4}$$

Other way:

$$\int_1^7\int_1^{}\left(2+\frac{1}{y}\right)dy\,dx \quad \leftarrow \text{from 1 to 4 will be } x \text{, from 4 to 7 will be } 8-x$$

$$\underset{\substack{\uparrow\\ \text{split into}\\ R_1 \text{ and } R_2}}{=} \underbrace{\int_1^4\int_1^x\left(2+\frac{1}{y}\right)dy\,dx}_{R_1} + \underbrace{\int_4^7\int_1^{8-x}\left(2+\frac{1}{y}\right)dy\,dx}_{R_2} =$$

= compute ....

$$= \int_1^4\left(2y+\ln|y|\right)\Bigg|_1^x dx + \int_4^7\left(2y+\ln|y|\right)\Bigg|_1^{8-x} dx =$$

$$= \int_1^4\left(2x+\ln x-2\right)dx + \int_4^7\left(2(8-x)+\ln(8-x)-2\right)dx =$$

$$= \int_1^4 2x\,dx + \int_1^4 \ln x\,dx - \int_1^4 2\,dx + \int_4^7 14\,dx - \int_4^7 2x\,dx + \int_4^7 \ln(8-x)\,dx =$$

$$= x^2\Big|_1^4 + x\ln x\Big|_1^4 - \int_1^4 x\cdot\frac{1}{x}\,dx - 2x\Big|_1^4 +$$

$$+ 14x\Big|_4^7 - x^2\Big|_4^7 + x\ln(8-x)\Big|_4^7 - \int_4^7 \frac{1}{8-x}(-1)\cdot x\,dx =$$

$$= 4^2 - 1^2 + 4\ln 4 - 0 - x\Big|_1^4 - 2(4-1) + 14(7-4) -$$

$$- 7^2 + 4^2 + 0 - 4\ln 4 - \int_4^7 \left(1 - \frac{8}{8-x}\right)dx =$$

$$= 15 - \left(x + 8\ln(8-x)\right)\Big|_4^7 =$$

$$= 15 - \left(7 + 0 - (4 + 8\ln 4)\right) =$$

$$= \boxed{12 + 8\ln 4}$$

First way was simpler!