(1) triple integrals

Given a region in ~~space~~ (collection of (x, y, z))

$\rightsquigarrow$ triple integral $\iiint\limits_{R} f(x,y,z)\, dV$, region in space $\to R$

where ~~Volume element~~ $dV = dx\,dy\,dz$

(in any 6 possible orders!)

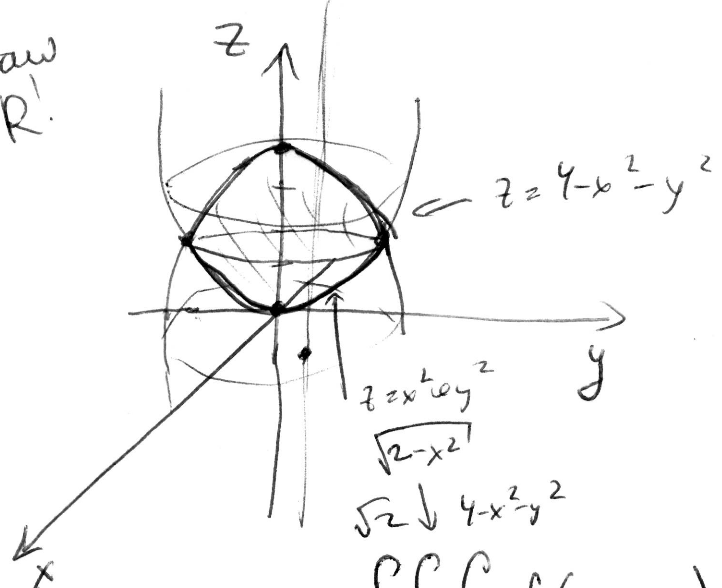
How to set up boundaries in $\iiint$?

~~Example~~ (1) $\iiint\limits_{R} f(x,y,z)\, dV$ over the region R

between the paraboloids $z = x^2+y^2$
and $z = 4-x^2-y^2$

equations of boundary of R!

Draw R!



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\boxed{x^2+y^2}}^{4-x^2-y^2} f(x,y,z)\, dz\, dy\, dx$$

1) $\uparrow$ for fixed x and y, range of z

$z = x^2+y^2$ (bottom face), $z = 4-x^2-y^2$ (top face)

② What values of (x,y) to consider?

Look at the "shadow" of R in xy-coordinate = portion of xy-plane above (or below) which R lives.

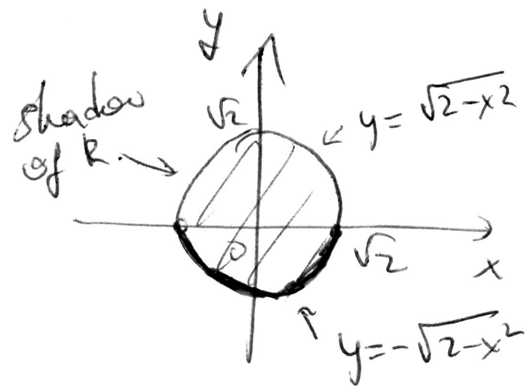
For which (x,y)  is above  ?

$z=4-x^2-y^2$ $\qquad$ $z=x^2+y^2$

$$4-x^2-y^2 \geq x^2+y^2$$

$$\Updownarrow$$

$x^2+y^2 \leq 2 \iff$ disk of radius $\sqrt{2}$ centered at $(0,0)$ in (x,y)-plane.



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} f(x,y,z)\,dz\,dy\,dx.$$

<u>Application:</u>

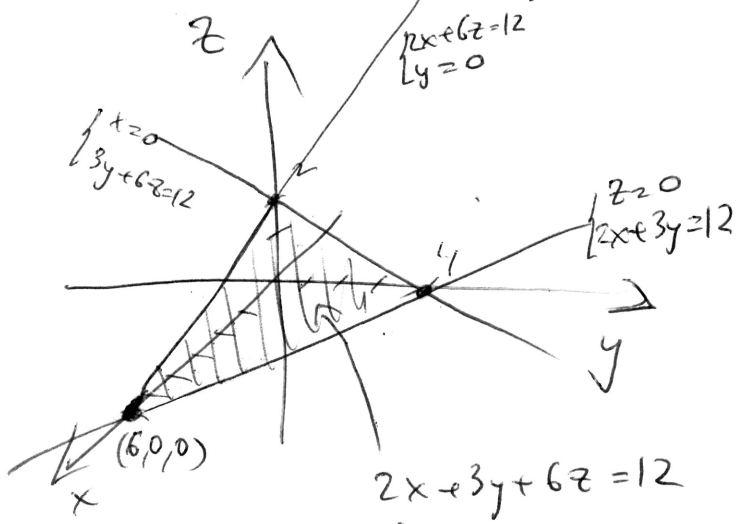
$$\text{volume}\underset{\substack{\uparrow\\ \text{space}}}{(R)} = \iiint_R 1\,dV$$

Examples

3) Find the volume of the solid R in the first octant bounded by the plane $2x+3y+6z=12$ and coordinate planes

Solution: volume $(R) = \iiint_R 1\, dV$

1) Draw the region R



$2x+3y+6z=12$

$$\left(\begin{array}{l} z=0 \to 2x+3y=12 \\ y=0 \to 2x+6z=12 \\ x=0 \to 3y+6z=12 \end{array}\right)$$

$2x+3y+6z=12$

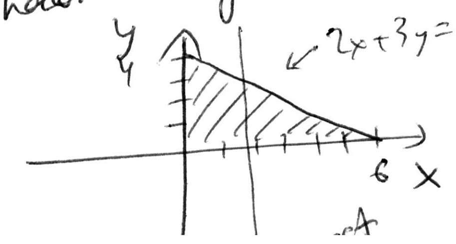
$$\left(z = 2 - \frac{1}{3}x - \frac{1}{2}y\right)$$

2) Set boundaries

$$\text{volume}(R) = \underbrace{\int_0^6 \int_0^{4-\frac{2}{3}x}}_{\text{"shadow" of } R \text{ in } (x,y) \text{ plan}} \underbrace{\int_0^{2-\frac{1}{3}x-\frac{1}{2}y}}_{x,y\text{ - fixed, where } z \text{ varies?}} 1\, dz\, dy\, dx$$

Draw shadow of R

$2x+3y=12 \quad \left(y = 4 - \frac{2}{3}x\right)$



$$\int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{2-\frac{1}{3}x-\frac{1}{2}y} 1 \, dz\, dy\, dx =$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} z \Big|_{z=0}^{z=2-\frac{1}{3}x-\frac{1}{2}y} dy\, dx =$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} \left(2-\frac{1}{3}x-\frac{1}{2}y\right) dy\, dx =$$

$$= \int_0^6 \left(2y-\frac{1}{3}xy-\frac{1}{4}y^2\right)\Big|_{y=0}^{y=4-\frac{2}{3}x} dx =$$

$$= \int_0^6 \left(2\left(4-\frac{2}{3}x\right)-\frac{1}{3}x\left(4-\frac{2}{3}x\right)-\frac{1}{4}\left(4-\frac{2}{3}x\right)^2\right) dx =$$

$$= \int_0^6 \left(8-\frac{4}{3}x-\frac{4}{3}x+\frac{2}{9}x^2-4+\frac{4}{3}x-\frac{1}{9}x^2\right) dx$$

$$= \int_0^6 \left(4-\frac{4}{3}x+\frac{1}{9}x^2\right) dx =$$

$$= \left(4x-\frac{2}{3}x^2+\frac{1}{27}x^3\right)\Big|_0^6 = 4\cdot 6-\frac{2}{3}6^2+\frac{1}{27}6^3-0$$

$$= 24 - \frac{2\cdot 2\cdot 3\cdot 2\cdot \cancel{3}}{\cancel{3}} + \frac{2^3\cdot \cancel{3^3}}{\cancel{27}} = 24-24+8$$

$$= \boxed{8}$$

(7)

Example 3 Compute the volume of R in the previous Example (Example 1)

Solution: $f(x,y,z) = 1$

$$\text{Volume}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1 \, dz \, dy \, dx$$

1) $$\int_{x^2+y^2}^{4-x^2-y^2} 1 \, dz = z \Big|_{z=x^2+y^2}^{z=4-x^2-y^2} = 4 - x^2 - y^2 - x^2 - y^2 = $$
$$= 4 - 2x^2 - 2y^2$$

then gets more complicated

$$\text{volume}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (4 - 2x^2 - 2y^2) \, dy \, dx$$

Can simplify? We can use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\text{volume}(R) = \int_0^{2\pi} \int_0^{\sqrt{2}} (4 - 2r^2) \, r \, dr \, d\theta = \ldots$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (4r - 2r^3) \, dr \, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{2} r^4\right) \Big|_{r=0}^{r=\sqrt{2}} d\theta = \int_0^{2\pi} (4-2) \, d\theta =$$

$$= 2\theta \Big|^{2\pi} = \boxed{4\pi}$$

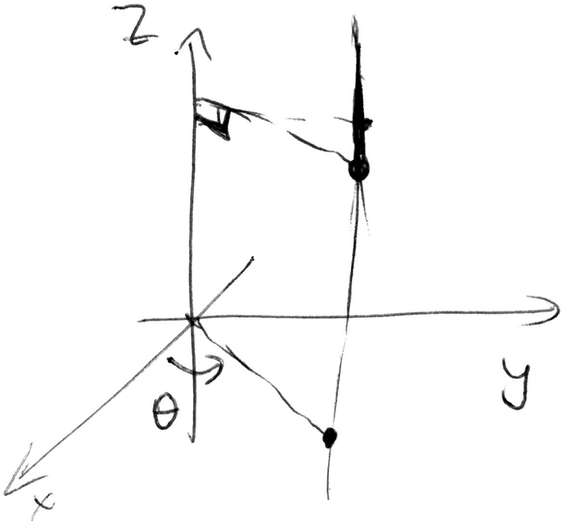
(4) Cylindrical coordinates

- describe point in space by (r, θ, z)

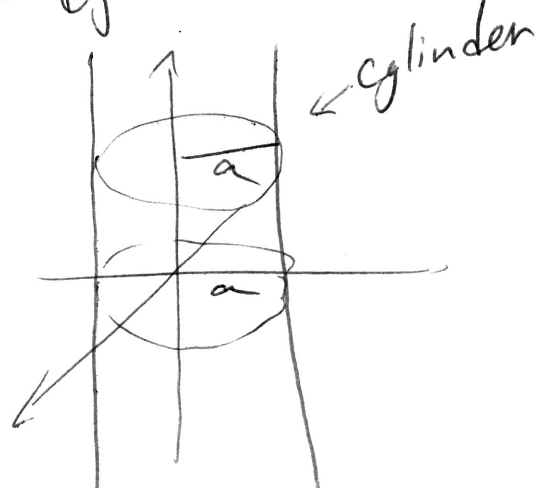
where $r =$ distance to z-axis $= \sqrt{x^2+y^2}$

Connection to Euclidean:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



What is $r=a$? in cylindrical coordinates

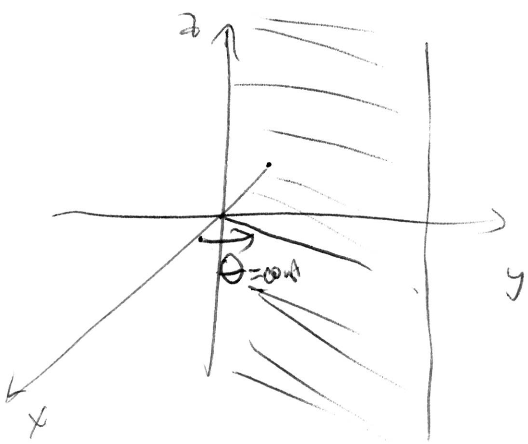


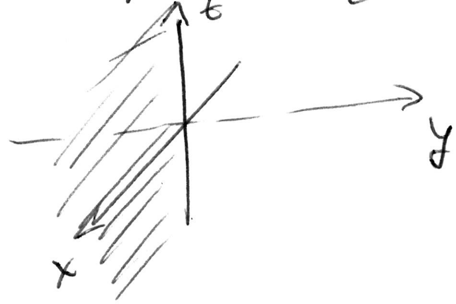
What is $\theta = \text{const}$? in cylindrical coordinates

How to think about θ=? "Axes z to which you have attached door"

Vertical half plane

For example if $\theta=0$, then get $\begin{cases} x \geq 0 \\ y = 0 \\ z \text{ - any number} \end{cases}$





6

Volume element:

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Usually, it is more convenient to use order

$$dV = dz\,(r\,dr\,d\theta)$$

as in the example.

Example 3

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} 1\,dz\,(r\,dr\,d\theta)$$

Applications:

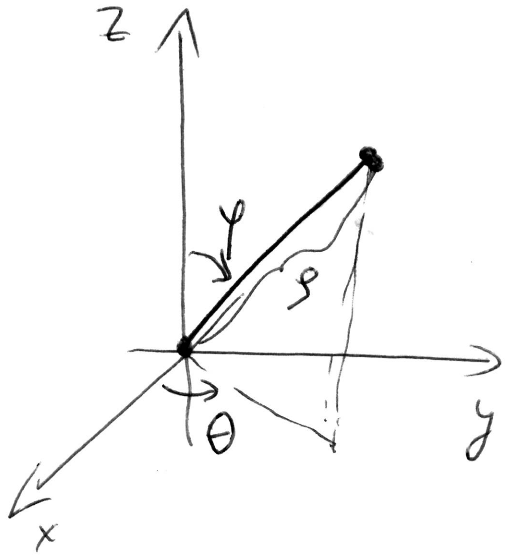
a) $\text{volume}(R) = \iiint_R 1\,dV$

b) $\text{mass}(R) = \iiint_R \rho\,dV$ where $\rho = \rho(x,y,z)$ - density

c) average value of f:

$$\bar{f} = \frac{1}{\text{volume}(R)} \iiint_R f(x,y,z)\,dV$$

(7) Spherical coordinates.



ρ = rho = distance to origin (≥ 0)

φ = phi = angle down from z-axis (0 to π)

θ = same as before (angle counterclockwise from positive xz-plane) (0 to 2π or $-\pi$ to π ...)

Connection to polar:

$$\begin{cases} z = \rho\cos\varphi \\ r = \rho\sin\varphi \\ \theta = \theta \end{cases}$$

where $\rho = \sqrt{r^2+z^2}$

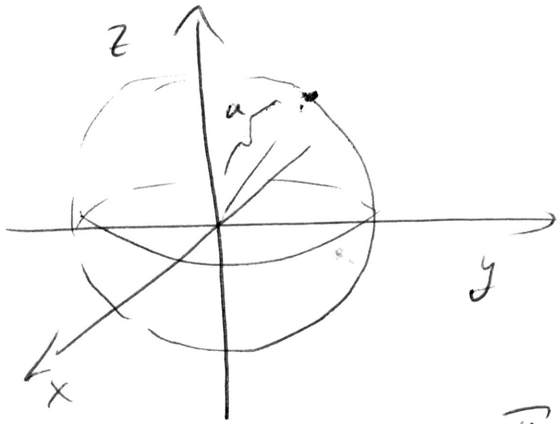
Connection to Euclidean:

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

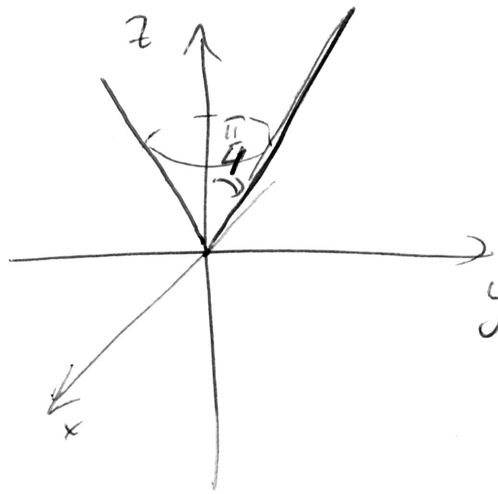
where $\rho = \sqrt{x^2+y^2+z^2}$

Note $\rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$

(8) What is $\rho = a$? Sphere of radius a centered at the origin



What is $\varphi = \frac{\pi}{4}$?



Cone whose generating lines make angle $\frac{\pi}{4}$ with z-axis

$\left(\varphi = \frac{\pi}{4} \to \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \to \right.$

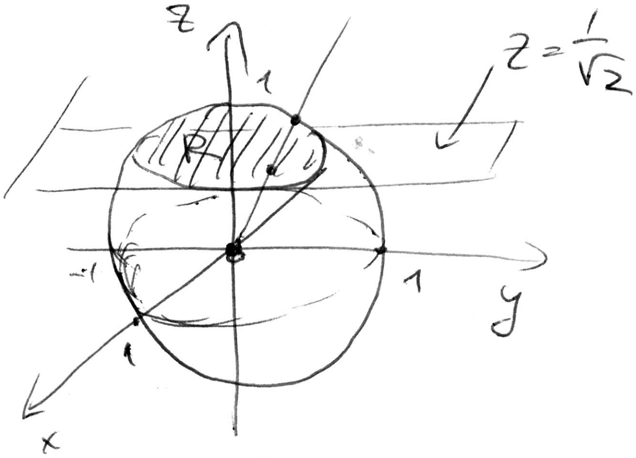
$\left. \to \underbrace{z = r = \sqrt{x^2+y^2}}_{\text{equation of cone}} \right)$

What is $\varphi = \frac{\pi}{2}$? It is xy-plane!

What is volume element dV in polar coordinates?

$$dV = \underbrace{dx\,dy\,dz}_{\text{Euclidean}} = \underbrace{r\,dz\,dr\,d\theta}_{\text{polar}} = \underbrace{\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}_{\text{spherical}}$$

(9) Example Compute the volume of portion of unit ball above $z = \frac{1}{\sqrt{2}}$.



use spherical coordinates (as we have part of ball)

$$\text{volume}(R) = \iiint_R 1\, dV =$$

for ρ_{max} leave sphere of radius 1

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}}\cdot\frac{1}{\cos\varphi}}^{1} 1 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

hit plane $z = \frac{1}{\sqrt{2}}$ for ρ_{min}

Sphere of radius 1 centered at the origin: $\rho = 1$

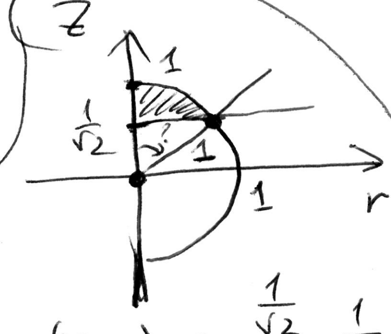
Plane $z = \frac{1}{\sqrt{2}}$ in spherical coordinates:

$$\rho\cos\varphi = \frac{1}{\sqrt{2}} \rightarrow \rho = \frac{1}{\sqrt{2}} \cdot \frac{1}{\cos\varphi}$$

(used that $z = \rho\cos\varphi$)

Boundaries for φ? Fix θ $\rightarrow$ consider half plane $\rightarrow$

Geometry



$$\cos(\varphi_{max}) = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}$$

$$\rightarrow \varphi_{max} = \frac{\pi}{4}$$

Other way to find intersection of plane and sphere:

$$\begin{cases} \rho = 1 \\ \rho\cos\varphi = \frac{1}{\sqrt{2}} \end{cases} \rightarrow \begin{cases} \rho = 1 \\ \cos\varphi = \frac{1}{\sqrt{2}} \end{cases} \rightarrow \begin{cases} \rho = 1 \\ \varphi = \frac{\pi}{4} \end{cases}$$

(10) $$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}}\frac{1}{\cos\varphi}}^{1} 1\cdot\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}\cdot\frac{1}{\cos\varphi}}^{1} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

1) $$\int_{\frac{1}{\sqrt{2}}\cdot\frac{1}{\cos\varphi}}^{1} \rho^2\sin\varphi\, d\rho = \frac{\rho^3}{3}\sin\varphi \Bigg|_{\rho=\frac{1}{\sqrt{2}}\cdot\frac{1}{\cos\varphi}}^{\rho=1} = \frac{1}{3}\sin\varphi - \frac{1}{6\sqrt{2}}\frac{\sin\varphi}{\cos^3\varphi}$$

2) $$\int_0^{\pi/4}\left(\frac{1}{3}\sin\varphi - \frac{1}{6\sqrt{2}}\cdot\frac{\sin\varphi}{\cos^3\varphi}\right)d\varphi =$$

$$= -\frac{1}{3}\cos\varphi - \frac{1}{12\sqrt{2}}\cdot\frac{1}{\cos^2\varphi}\Bigg|_{\varphi=0}^{\varphi=\frac{\pi}{4}} = \dots = \frac{1}{3} - \frac{5}{12\sqrt{2}}$$

$$\left(\int\frac{\sin\varphi}{\cos^3\varphi}d\varphi \underset{\uparrow}{=} -\int\frac{du}{u^3} = -\frac{u^{-2}}{-2} = \frac{1}{2u^2} = \frac{1}{2\cos^2\varphi}\right.$$

$u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos(0) = 1$)

3) $$\int_0^{2\pi}\left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)d\theta = \left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)\theta\Bigg|_{\theta=0}^{\theta=2\pi} =$$

$$= \left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)\cdot 2\pi = \boxed{\frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}}$$

f is continuous

(11) Change order of integration in

$$\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx$$

to $\underbrace{\int\int\int}_{\text{what are boundaries}} f(x,y,z)\,dy\,dx\,dz$

Solution: Change 2 integrals at a time

$$\int_0^1\int_0^x\int_0^y f\,dz\,dy\,dx = \int_0^1\int_0^x\int_z^x f\,dy\,dz\,dx =$$

$$= \int_0^1\int_z^1\int_z^x f\,dy\,dx\,dz$$