MATH 2177

Lecture 1

Introduce myself.

Why take Math 2177? It's required for your major!

It's fun!

It's a quick journey into 4 topics!

Final 30% December 10

- Multivariable Integral Calculus (20% of grade, Midterm 1 - September 28)
- Matrices & systems of linear equations (20% of grade, Midterm 2 - October 24)
- 2nd order constant coefficient O.D.E. (20% - Midterm 3 - November 19)
- Fourier Series & P.D.E.

It's powerful! You will be able to apply your new knowledge for your major!

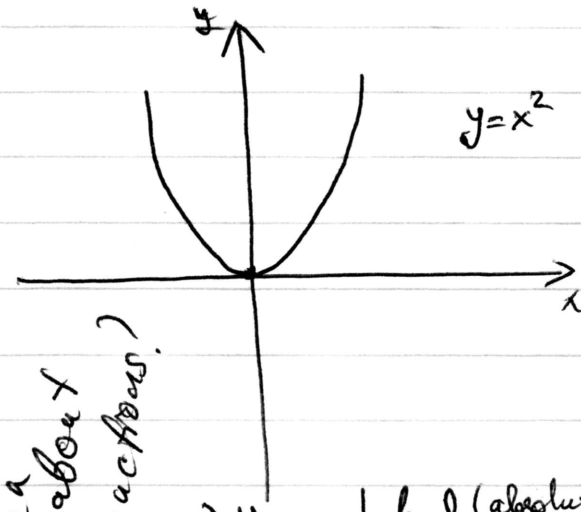
Syllabus on Canvas & my website u.osu.edu/erchenko.1

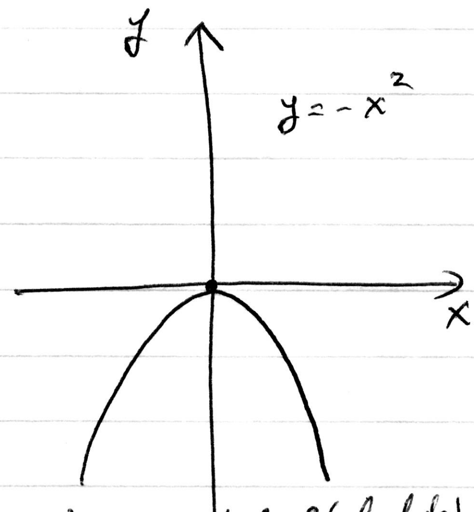
10% Homework → every Tuesday!

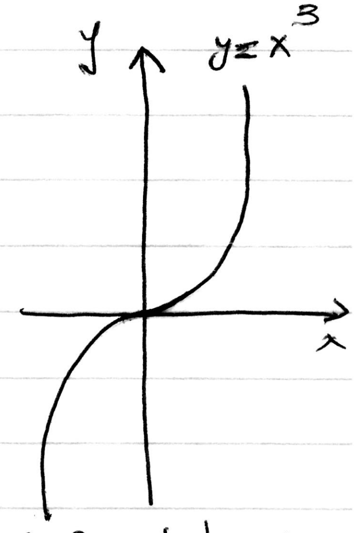
solutions for homework

Lecture notes.

Journey 1 begins!







What can you say about these functions?

1) Has global (absolute) minimum at $x=0$

2) Derivative at $x=0$:
$y'(x)=2x \to y'(0)=\underline{\underline{0}}$

3) Second derivative at $x=0$:
$y''(x)=2 \to y''(0)=2>0$

1) Has global (absolute) maximum at $x=0$

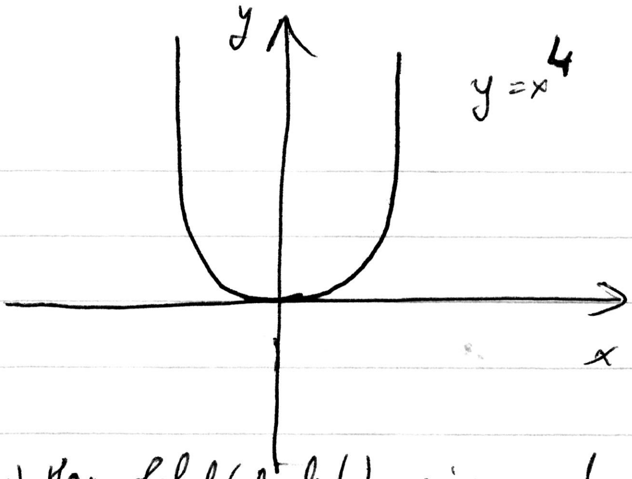
2) $y'(x)=-2x \to y'(0)=\underline{\underline{0}}$

3) $y''(x)=-2 \to$
$\to y''(0)=-2<0$

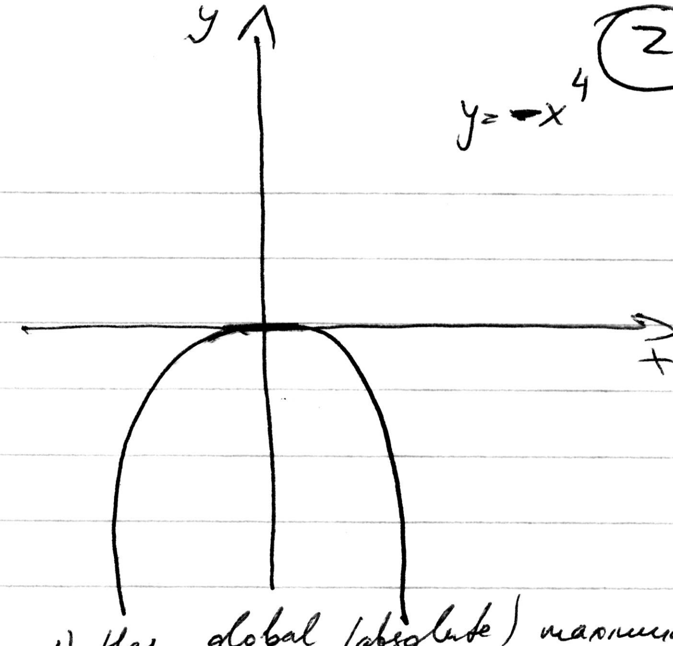
1) Doesn't have local/global minimum or maximum at $x=0$

2) $y'(x)=3x^2 \to y'(0)=\underline{\underline{0}}$

3) $y''(x)=6x \to y''(0)=0$



1) Has global (absolute) minimum at $x=0$
2) $y'(x) = 4x^3 \to y'(0) = 0$
3) $y''(x) = 12x^2 \to y''(0) = 0$



1) Has global (absolute) maximum at $x=0$
2) $y'(x) = -4x^3 \to y'(0) = 0$
3) $y''(x) = -12x^2 \to y''(0) = 0$

$f(x)$	$f(x,y)$
Critical	**point**
Definition An interior point $x=a$ in the domain of f is a critical point of f if either 1) $f'(a) = 0$, or 2) The derivative of f at $x=a$ does not exist	An interior point $(x,y) = (a,b)$ in the domain of f is a critical point of f if either 1) $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle = \langle 0, 0 \rangle$ (i.e, $f_x(a,b)=0$ and $f_y(a,b)=0$), or 2) At least one of the partial derivatives f_x and f_y does not exist at (a,b).

Example

Find critical point of the following function

$f(x) = x^2$

Solution: $f'(x) = 2x$ ← defined everywhere

Critical point: $f'(x) = 0 \iff 2x = 0 \iff x = 0$

$x = 0$ – critical point.

$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

Solution

$\nabla f(x,y) = \langle f_x, f_y \rangle =$

$= \langle 2x - 2y + 2, -2x + 6y - 2 \rangle$ ← defined everywhere

Critical point: $\nabla f(x,y) = \langle 0, 0 \rangle$

$\begin{cases} 2x - 2y + 2 = 0 \\ -2x + 6y - 2 = 0 \end{cases} \iff \begin{cases} x - y = -1 \\ -x + 3y = 1 \end{cases} \iff \begin{cases} x = -1 \\ y = 0 \end{cases} \iff (-1, 0)$ – critical point.

$f(x)$ | $f(y)$

What type of critical point?

Local maximum point

$f(x)$ has local maximum value at $x=a$ if $f(x) \le f(a)$ for any point x in some neighborhood of $x=a$. $x=a$ - local maximum point

$f(x,y)$ has local maximum value at $(x,y)=(a,b)$ if $f(x,y) \le f(a,b)$ for any point (x,y) in some neighborhood of $(x,y)=(a,b)$. $(x,y)=(a,b)$ - local maximum point

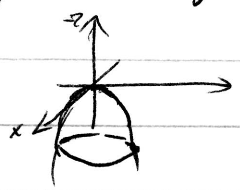
Example

$f(x) = -x^2$

$x=0$ - local maximum point

$(f'(x) = -2x \to f'(0) = 0)$

$f(x,y) = -x^2 - y^2$



$(\nabla f = \langle -2x, -2y \rangle \to \nabla f(0,0) = \langle 0,0 \rangle)$

$(0,0)$ - local maximum point

Local minimum point

$f(x)$ has local minimum value at $x=a$ if $f(x) \ge f(a)$ for any point x in some neighborhood of $x=a$, $x=a$ - local minimum point

$f(x,y)$ has local minimum value at $(x,y)=(a,b)$ if $f(x,y) \ge f(a,b)$ for any point (x,y) in some neighborhood of $(x,y)=(a,b)$. (a,b) - local minimum point

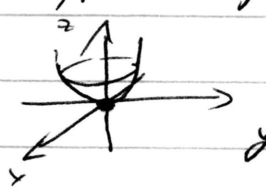
Example

$f(x) = x^2$

$x=0$ - local minimum point

$(f'(x) = 2x \to f'(0) = 0)$

$f(x,y) = x^2 + y^2$



$(0,0)$ - local minimum point

$(\nabla f = \langle 2x, 2y \rangle \to \nabla f(0,0) = \langle 0,0 \rangle)$

[redacted]

Theorem

If f has local ~~max~~/min value at $x=a$ and [redacted] f' exists at $x=a$, then $f'(a)=0$

Theorem

If f has local max/min value at $(x,y)=(a,b)$ and the partial derivatives f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$ $\nabla f(a,b) = \langle 0,0 \rangle$

converse of theorem is not true!

$f(x)$	$f(x,y)$
Neither maximum/minimum	**Saddle point**
$f(x)$ doesn't have max/min at $x=a$ if, in every neighborhood of $x=a$, there are points x for which $f(x) > f(a)$ and point for which $f(x) < f(a)$	$f(x,y)$ has a saddle point at (a,b) if, in every neighborhood of (a,b), there are points (x,y) for which $f(x,y) > f(a,b)$ and point for which $f(x,y) < f(a,b)$

Example

$f(x) = -x^3$

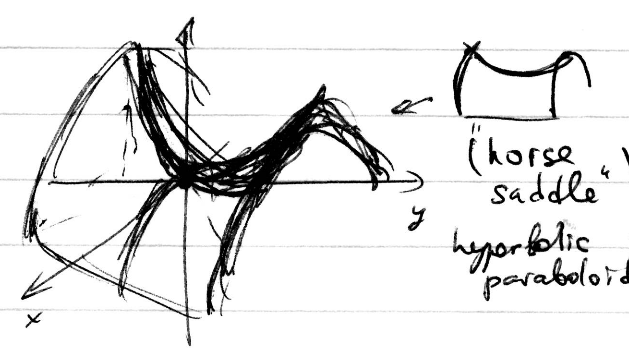
$x=0$ is a critical point but not a max/min

$f(x) = x^2 - y^2$

$\nabla f(x,y) = \langle 2x, -2y \rangle$

$\nabla f(0,0) = \langle 0, 0 \rangle$

$(0,0)$ – critical point, but not a max/min, it is saddle point



Example

What type of critical point is $(-1,0)$ for $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

<u>Solution:</u>

$f(-1,0) = (-1)^2 - 2(-1)\cdot 0 + 3\cdot 0^2 + 2\cdot(-1) - 2\cdot 0 = -1$

$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y =$

$= x^2 - 2xy + y^2 + 2y^2 + 2(x-y) =$

$= \underbrace{(x-y)^2 + 2(x-y) + 1} - 1 + 2y^2 =$

$= (x-y+1)^2 + 2y^2 - 1 \geq -1 = f(-1,0)$

$(-1,0)$ – minimum point

Goals

1) Find all critical points of a given function (can be several)
2) Determine types of each critical point
3) Find the absolute (global) max/min of a function

Is there sometimes easier way to determine the type of a critical point?

Yes! :)

Second derivative test

$f(x)$	$f(x,y)$
1) Second derivative of f exists in a neighborhood of $x=a$ + 2) at $x=a$ $f'(a)=0$	1) The second partial derivatives $(f_{xx}, f_{xy}, f_{yx}, f_{yy})$ of f exist and continuous in a neighborhood of (a,b) + 2) $\nabla f(a,b) = \langle 0, 0 \rangle$, i.e. $f_x(a,b)=0$ and $f_y(a,b)=0$.
	Denote the discriminant $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$
Then: a) If $f''(a)>0$, then f has local min at $x=a$ b) If $f''(a)<0$, then f has local max at $x=a$ c) If $f''(a)=0$, then the test is inconclusive	Then: a) If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then f has a local maximum at (a,b) b) If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then f has a local min at (a,b) c) If $D(a,b)<0$, then f has a saddle point at (a,b) d) If $D(a,b)=0$, then the test is inconclusive.

Example $f(x,y) = 2x^4 + y^4 \to \begin{matrix} f_x = 8x^3 \\ f_y = 4y^3 \end{matrix} \to$ To have $\nabla f = 0$, we have $\begin{cases} 8x^3 = 0 \\ 4y^3 = 0 \end{cases} \to \begin{cases} x = 0 \\ y = 0 \end{cases}$

$(0,0)$ - critical point. $f_{xx} = 24x^2$, $f_{xy} = 0 = f_{yx}$, $f_{yy} = 12y^2 \to D(0,0) = 0 \to$ the test is inconclusive

Example

Find and classify all the critical points for

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Solution: 1. Find critical points,

$$f_x(x,y) = 6xy - 6x = 6x(y-1)$$

$$f_y(x,y) = 3x^2 + 3y^2 - 6y$$

exist

(a,b) is a critical point if and only if $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle = \langle 0,0 \rangle$

Solve system:

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \iff \begin{cases} 6x(y-1) = 0 & (1) \\ 3x^2 + 3y^2 - 6y = 0 & (2) \end{cases}$$

Solution of (1): $6x(y-1) = 0$

$x = 0$

satisfy (2) ↓

$3 \cdot 0^2 + 3y^2 - 6y = 0$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0$ $\quad$ $y = 2$

$y - 1 = 0$

$y = 1$

↓ satisfy (2)

$3x^2 + 3 \cdot 1^2 - 6 \cdot 1 = 0$

$3x^2 - 3 = 0$

$x^2 = 1$

$x = 1$ $\quad$ $x = -1$

How many critical points we have? 4

Critical points: $(0,0)$, $(0,2)$, $(1,1)$, $(-1,1)$

2. Classify critical points.

Compute second partial derivatives:

$$\left.\begin{aligned} f_{xx}(x,y) &= 6y-6 \\ f_{yy}(x,y) &= 6y-6 \\ f_{xy}(x,y) &= f_{yx}(x,y) = 6x \end{aligned}\right\}$$

exists and continuous.

↓

Apply second derivative test to classify critical points!

$$D(x,y) = f_{xx}(x,y)\cdot f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$$

a) $(x,y) = (0,0)$:

$$f_{xx}(0,0) = 6\cdot 0 - 6 = -6$$
$$f_{yy}(0,0) = 6\cdot 0 - 6 = -6$$
$$f_{xy}(0,0) = 6\cdot 0 = 0$$

$$\left.\begin{aligned} D(0,0) &= -6\cdot(-6) - 0^2 = 36 > 0 \\ f_{xx}(0,0) &= -6 < 0 \end{aligned}\right\} \to \text{local max at } (0,0)$$

b) $(x,y) = (0,2)$:

$$\left.\begin{aligned} f_{xx}(0,2) &= 6 \\ f_{yy}(0,2) &= 6 \\ f_{xy}(0,2) &= 0 \end{aligned}\right\} \to \left.\begin{aligned} D(0,2) &= 36 > 0 \\ f_{xx}(0,2) &= 6 > 0 \end{aligned}\right\} \to \text{local min at } (0,2)$$

c) $(x,y) = (1,1)$:

$$\left.\begin{aligned} f_{xx}(1,1) &= 0 \\ f_{yy}(1,1) &= 0 \\ f_{xy}(1,1) &= 6 \end{aligned}\right\} \Rightarrow D(1,1) = -36 < 0 \} \to \text{saddle at } (1,1)$$

d) $(x,y) = (-1,1)$:

$$\left.\begin{aligned} f_{xx}(-1,1) &= 0 \\ f_{yy}(-1,1) &= 0 \\ f_{xy}(-1,1) &= -6 \end{aligned}\right\} \to D(-1,1) = -36 < 0 \to \text{saddle at } (-1,1)$$