Feasible-Directions Method: Solution algorithms

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What

Feasible-Directions Method

Feasible-Directions Method

Feasible-Directions Method

The benefit of the feasible-directions method is that it ensures that each point found is feasible in the original problem constraints.

Thus, there is no need to adjust penalty weights to ensure feasibility.

The feasible-directions method is limited, however, as it can only easily be applied to problems with relatively simple constraints.

Feasible-Directions Method

In this discussion, we take the case of problems with linear constraints.

Thus, without loss of generality, we assume that we have a linearly constrained nonlinear optimization problem of the form:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & f(x) \\ \text{s.t.} \ & Ax - b \leq 0, \end{aligned}$$

Feasible-Directions Method

$$\min_{x \in \mathbb{R}^n} \; f(x)$$
$$\text{s.t.} \quad Ax - b \leq 0,$$

Feasible-Directions Method

where A is an $m \times n$ matrix of constraint coefficients and b is an $m \times 1$ vector of constants.

This problem structure only has inequality constraints.

Feasible-Directions Method

Note, however, that an equality constraint of the form:

$$h(x) = 0,$$

can be written as two inequalities:

$$h(x) \leq 0,$$

and:

$$h(x) \geq 0.$$

Feasible-Directions Method

Note, however, that converting equality constraints into inequalities in this way can create numerical difficulties in practice.

However, the problem structure that we assume is generic in that the feasible-directions method can be applied to a problem with any combination of linear equality and inequality constraints.

Feasible-Directions Method

The feasible-directions method follows the same approach as the Generic Algorithm for Unconstrained Nonlinear Optimization Problems, with four major differences.

Feasible-Directions Method

First, the starting point x^0, must be feasible in the constraints.

Feasible-Directions Method

Second, the updating formula used in each iteration has the form:

$$x^{k+1} \leftarrow x^k + \alpha^k \cdot (d^k - x^k).$$

As seen in the following discussion, this updating formula is used because it allows us to easily ensure that the point that we find after each iteration is feasible.

Feasible-Directions Method

Third, we must also change our direction and line searches conducted in each iteration to ensure that the new point found after each iteration is feasible.

Feasible-Directions Method

Fourth, we must also modify our termination criteria. This is because when constraints are included in a problem, a point being stationary is no longer necessary for it to be a local minimum.

Feasible-Directions Method

We now provide details on how to find a feasible starting point and how the search direction and line search are modified in the feasible-directions method.

Finding a Feasible Starting Point

Finding a Feasible Starting Point

Finding a feasible starting point means finding a point, x^0, that satisfies the constraints of the problem.

In other words, we want to find a vector x^0 such that $Ax^0 - b \leq 0$. We can do this by solving the linear optimization problem:

Finding a Feasible Starting Point

$$\min_{x \in \mathbb{R}^n} \; 1^\top x$$

$$\text{s.t.} \quad Ax - b \leq 0$$

Finding a Feasible Starting Point

We arbitrarily set the objective function of this problem to $1^\top x$. This is because the sole purpose is to find a feasible starting point for the feasible-directions method.

We are not concerned with what feasible point we begin the feasible-directions method from. Indeed, we do not even need to solve this problem to optimality. As soon as we have we have a feasible solution, we can terminate at that point.

Finding a Feasible Search Direction

Finding a Feasible Search Direction

We find a feasible search direction by following the same underlying logic as in the steepest descent method.

Namely, we want to find a direction that gives the greatest improvement in the objective function.

Finding a Feasible Search Direction

As in the derivation of the search directions for unconstrained nonlinear problems, we ignore the step size parameter when conducting the direction search.

Thus, in the following discussion we assume that $\alpha^k = 1$ and that our updated point has the form:

$$x^{k+1} = x^k + (d^k - x^k).$$

Later, when we conduct the line search, we reintroduce the step size parameter.

$$x^{k+1} = x^k + (d^k - x^k)$$

Finding a Feasible Search Direction

Thus, we can write a first-order Taylor approximation of the objective-function value at the new point as:

$$f(x^{k+1}) = f(x^k + (d^k - x^k)) \approx \boxed{f(x^k) + (d^k - x^k)^\top \nabla f(x^k)}$$

$$x^{k+1} = x^k + (d^k - x^k)$$

Finding a Feasible Search Direction

Our goal is to choose d^k to minimize this Taylor approximation.

Because we have the linear constraints in our problem, however, we want to also choose d^k in such a way that guarantees that the new point is feasible.

We do this by imposing the constraints:

$$Ax^{k+1} - b = A \cdot (x^k + (d^k - x^k)) - b = \boxed{Ad^k - b \leq 0.}$$

Finding a Feasible Search Direction

Thus, in the feasible-directions method we always find our search direction by solving the following linear optimization problem, which is outlined in the Feasible-Directions-Search-Direction Rule.

Finding a Feasible Search Direction

Given a linearly-constrained nonlinear optimization problem and a current point, x^k, the **feasible-directions** search direction is found by solving the linear optimization problem:

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Finding a Feasible Search Direction

$$\begin{aligned} \min_{d^k \in \mathbb{R}^n} \quad & f(x^k) + (d^k - x^k)^\top \nabla f(x^k) \\ \text{s.t.} \quad & Ad^k - b \leq 0 \end{aligned}$$

Finding a Feasible Search Direction

It is important to stress that the search direction in the feasible-directions method is found in a very different way than the steepest-descent and Newton's method directions for unconstrained problems.

The steepest-descent and Newton's method directions are found by applying simple rules.

That is, those directions are found by simply using derivative information.

Finding a Feasible Search Direction

The feasible-directions method requires a linear optimization problem to be solved at each iteration to find the search direction.

This is typically easy to do, however, as the Simplex method can efficiently solve very large linear optimization problems.

Termination Criteria

Termination Criteria

The termination criteria with the feasible-directions is typically not the same as for an unconstrained problem. This is because with an unconstrained problem local minima are stationary points. Thus, iterative algorithms applied to unconstrained problems terminate at stationary points.

Local minima of constrained problems are not typically stationary points. Rather, they are points that satisfy the KKT condition, which involve the gradients of the objective function *and* constraints.

Termination Criteria

The termination criteria in the feasible-directions method depends on the value of the objective function from the search-direction problem.

Specifically, we examine the sign of the $(d^k - x^k)^\top \nabla f(x^k)$ term in the objective function.

Termination Criteria

Note that $d^k = x^k$ is feasible in the constraints.

Substituting $d^k = x^k$ into the objective function gives:

$$f(x^k)+(d^k-x^k)^\top \nabla f(x^k) = f(x^k)+(x^k-x^k)^\top \nabla f(x^k) = f(x^k).$$

Termination Criteria

We can conclude from this that when the search-direction problem is solved, the optimal value of

$$f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$

must be no greater than

$$f(x^k).$$

$$x^{k+1} = x^k + (d^k - x^k)$$

Termination Criteria

If:

$$f(x^k) + (d^k - x^k)^\top \nabla f(x^k) = f(x^k),$$

which means that:

$$(d^k - x^k)^\top \nabla f(x^k) = 0,$$

We look at the objective function of the new point

then the search-direction problem is not able to find a feasible direction to move in that improves the objective function. Thus, we terminate the feasible-directions method.

$$x^{k+1} = x^k + (d^k - x^k)$$

Termination Criteria

Otherwise, if:

$$f(x^k) + (d^k - x^k)^\top \nabla f(x^k) < f(x^k),$$

which means that:

$$(d^k - x^k)^\top \nabla f(x^k) < 0,$$

then the search-direction problem has found a feasible direction that improves the objective function and we continue the algorithm.

Conducting a Feasible Line Search

$$x^{k+1} = x^k + \alpha^k \cdot (d^k - x^k)$$

Conducting a Feasible Line Search

With the feasible-directions method, we can use either of the exact line search or other rule, so long as we restrict the step size to be less than or equal to one. If α^k is between zero and one, we can show that the new point is guaranteed to be feasible.

To see this, note that if the new point is feasible, that means that:

$$A \cdot (x^k + \alpha^k \cdot (d^k - x^k)) - b \leq 0.$$

$$x^{k+1} = x^k + \alpha^k \cdot (d^k - x^k)$$

Conducting a Feasible Line Search

By distributing the product and collecting terms, we can rewrite the left-hand side of this inequality as:

$$\begin{aligned} A \cdot (x^k + \alpha^k \cdot (d^k - x^k)) - b &= Ax^k + \alpha^k Ad^k - \alpha^k Ax^k - b \\ &= (1 - \alpha^k)Ax^k - (1 - \alpha^k)b + \alpha^k Ad^k - \alpha^k b \\ &= (1 - \alpha^k)(Ax^k - b) + \alpha^k (Ad^k - b). \end{aligned}$$

Conducting a Feasible Line Search

We next note that because our current point, x^k, is feasible, $Ax^k - b \leq 0$. Moreover, the constraint that we use to find the search direction, ensures that $Ad^k - b \leq 0$.

Combining these observations with the fact that $\alpha^k \geq 0$ and $1 - \alpha^k \geq 0$, tells us that:

$$A \cdot (x^k + \alpha^k (d^k - x^k)) - b = (1 - \alpha^k)(Ax^k - b) + \alpha^k (Ad^k - b) \leq 0,$$

meaning that x^{k+1} is feasible, so long as we have $0 \leq \alpha^k \leq 1$.

Conducting a Feasible Line Search

$$\begin{aligned} \min_{\alpha^k} \ & f(x^k + \alpha^k \cdot (d^k - x^k)) \\ \text{s.t.} \ & 0 \leq \alpha^k \leq 1 \end{aligned}$$

Example

Example

Consider the linearly-constrained nonlinear optimization problem:

$$
\begin{aligned}
\min_{x} \; & f(x) = (x_1 + 1/2)^2 + (x_2 - 2)^2 \\
\text{s.t.} \; & x_1 \leq 0 \\
& x_2 \leq 0 \\
& -x_1 - 10 \leq 0 \\
& -x_2 - 10 \leq 0.
\end{aligned}
$$

GAMS

```
variables z, x1, x2;
equations of, e1, e2, e3, e4;
of.. z =e= power((x1+1/2),2)+power((x2-2),2);
e1.. x1 =l= 0;
e2.. x2 =l= 0;
e3.. -x1-10 =l= 0;
e4.. -x2-10 =l= 0;
model fea /all/;
solve fea using nlp minimizing z;
option decimals = 8;
display x1.l, x2.l, z.l;
```

GAMS

```
----      11 VARIABLE x1.L                    =   -0.50000000
             VARIABLE x2.L                    =    0.00000000
             VARIABLE z.L                     =    4.00000000
```

Example

$$\min_x \ f(x) = (x_1 + 1/2)^2 + (x_2 - 2)^2$$
$$\text{s.t.} \quad x_1 \le 0$$
$$x_2 \le 0$$
$$-x_1 - 10 \le 0$$
$$-x_2 - 10 \le 0.$$

Feasible

Starting from the point $x^0 = (-1, -1)^\top$, we use the feasible-directions method with an exact line search to solve the problem.

Example

$$\begin{aligned}\min_{x}\ & f(x) = (x_1 + 1/2)^2 + (x_2 - 2)^2 \\ \text{s.t.}\ & x_1 \leq 0 \\ & x_2 \leq 0 \\ & -x_1 - 10 \leq 0 \\ & -x_2 - 10 \leq 0.\end{aligned}$$

To find a feasible direction to move away from x^0 in, we compute the gradient of the objective function, which is:

$$\nabla f(x) = \begin{pmatrix} 2(x_1 + 1/2) \\ 2(x_2 - 2) \end{pmatrix}.$$

$$\min_{d^k \in \mathbb{R}^n} \quad f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Example

Thus, we find the search direction at $x^0 = (-1, -1)^\top$ by solving the linear optimization problem:

$$\min_{d^0} \; f(x^0) + (d^0 - x^0)^\top \nabla f(x^0) = \frac{37}{4} - (d_1^0 + 1) - 6(d_2^0 + 1)$$
$$\begin{aligned} \text{s.t.} \quad & d_1^0 \leq 0 \\ & d_2^0 \leq 0 \\ & -d_1^0 - 10 \leq 0 \\ & -d_2^0 - 10 \leq 0. \end{aligned}$$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \;\; Ad^k - b \leq 0$$

Example

$$\min_{d^0} \; \frac{37}{4} - (d_1^0 + 1) - 6(d_2^0 + 1)$$
$$\text{s.t.} \;\; d_1^0 \leq 0$$
$$d_2^0 \leq 0$$
$$-d_1^0 - 10 \leq 0$$
$$-d_2^0 - 10 \leq 0.$$

Example

We have $d^0 = (0, 0)^\top$ as an optimal solution to this problem.

We further have that $(d^0 - x^0)^\top \nabla f(x^0) = -7$.

Thus, we indeed have a feasible direction that improves the objective function. This means that we should proceed with an iteration and not yet terminate the algorithm.

$$\min_{\alpha^k} \ f(x^k + \alpha^k \cdot (d^k - x^k))$$
$$\text{s.t.} \ \ 0 \leq \alpha^k \leq 1$$

Example

We conduct the exact line search by solving the problem:

$$\min_{\alpha^0} \ f(x^0 + \alpha^0 \cdot (d^0 - x^0)) = f\left(\begin{pmatrix} \alpha^0 - 1 \\ \alpha^0 - 1 \end{pmatrix}\right) = \left(\alpha^0 - \frac{1}{2}\right)^2 + (\alpha^0 - 3)^2$$
$$\text{s.t.} \ \ 0 \leq \alpha^0 \leq 1.$$

$$\min_{\alpha^k} \; f(x^k + \alpha^k \cdot (d^k - x^k))$$
$$\text{s.t.} \;\; 0 \leq \alpha^k \leq 1$$

Example

$$\min_{\alpha^0} \; \left(\alpha^0 - \frac{1}{2}\right)^2 + (\alpha^0 - 3)^2$$
$$\text{s.t.} \quad 0 \leq \alpha^0 \leq 1.$$

Example

If we solve:

$$\frac{\partial}{\partial \alpha^0} f(x^0 + \alpha^0 \cdot (d^0 - x^0)) = 0,$$

this gives $\alpha^0 = 7/4.$

However, we know that α^0 must be no greater than 1. If we substitute $\alpha^0 = 1$ into $f(x^0 + \alpha^0 \cdot (d^0 - x^0))$, we find that this gives the optimal step size. Thus, we have $\alpha^0 = 1$ and our new point after one iteration is $x^1 = (0, 0)^\top.$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \;\; Ad^k - b \leq 0$$

Example

To conduct another iteration of the feasible-directions method from our new point, x^1, we find the search direction by solving the following linear optimization problem:

$$\min_{d^1} \; f(x^1) + (d^1 - x^1)^\top \nabla f(x^1) = \frac{17}{4} + d_1^1 - 4d_2^1$$
$$\text{s.t.} \;\; d_1^1 \leq 0$$
$$d_2^1 \leq 0$$
$$-d_1^1 - 10 \leq 0$$
$$-d_2^1 - 10 \leq 0.$$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \; Ad^k - b \le 0$$

Example

$$\begin{aligned}
\min_{d^1} \; & \frac{17}{4} + d_1^1 - 4d_2^1 \\
\text{s.t.} \; & d_1^1 \le 0 \\
& d_2^1 \le 0 \\
& -d_1^1 - 10 \le 0 \\
& -d_2^1 - 10 \le 0.
\end{aligned}$$

Example

We have

$d^1 = (10, 0)^\top$ as an optimal solution and

$(d^1 - x^1)^\top \nabla f(x^1) = -10.$

$$\min_{\alpha^k} \ f(x^k + \alpha^k \cdot (d^k - x^k))$$
$$\text{s.t.} \ \ 0 \leq \alpha^k \leq 1$$

Example

Thus, we must continue conducting another iteration of the feasible-directions method, by solving the exact-line-search problem:

$$\min_{\alpha^1} \ f(x^1 + \alpha^1 \cdot (d^1 - x^1)) = f\left(\begin{pmatrix} -10\alpha^1 \\ 0 \end{pmatrix}\right) = \left(-10\alpha^1 + \frac{1}{2}\right)^2 + (-2)^2$$
$$\text{s.t.} \ \ 0 \leq \alpha^1 \leq 1.$$

$$\min_{\alpha^k} f(x^k + \alpha^k \cdot (d^k - x^k))$$
$$\text{s.t.} \quad 0 \le \alpha^k \le 1$$

Example

$$\min_{\alpha^1} \left(-10\alpha^1 + \frac{1}{2}\right)^2 + (-2)^2$$
$$\text{s.t.} \quad 0 \le \alpha^1 \le 1.$$

Example

Solving:

$$\frac{\partial}{\partial \alpha^1} f(x^1 + \alpha^1 \cdot (d^1 - x^1)) = 0,$$

gives $\alpha^1 = 1/20.$ We can further verify that:

$$\frac{\partial^2}{\partial {\alpha^1}^2} f(x^1 + \alpha^1 \cdot (d^1 - x^1)) = 200,$$

which is positive definite, meaning that $\alpha^1 = 1/20$ is a global minimum.

Thus, our new point is $x^2 = (-1/2, 0)^\top.$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Example

To conduct a third iteration of the feasible-directions method starting from x^2, we find the search direction by solving the following linear optimization problem:

$$\begin{aligned}
\min_{d^2} \; & f(x^2) + (d^2 - x^2)^\top \nabla f(x^2) = 4 - 4d_2^2 \\
\text{s.t.} \; & d_1^2 \leq 0 \\
& d_2^2 \leq 0 \\
& -d_1^2 - 10 \leq 0 \\
& -d_2^2 - 10 \leq 0.
\end{aligned}$$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Example

$$\begin{aligned}
\min_{d^2} \; & 4 - 4d_2^2 \\
\text{s.t.} \; & d_1^2 \leq 0 \\
& d_2^2 \leq 0 \\
& -d_1^2 - 10 \leq 0 \\
& -d_2^2 - 10 \leq 0.
\end{aligned}$$

Example

We have $d^2 = (d_1^2, 0)^\top$, where d_1^2 can be any value between -10 and 0 as an optimal solution.

Moreover, we find that $(d^2 - x^2)^\top \nabla f(x^2) = 0$ (regardless of the value of d_1^2 chosen), meaning that we are not able to find a feasible direction to move in that improves the objective function. Thus, we should terminate the algorithm with $x^2 = (-1/2, 0)^\top$ as our final solution.

Unbounded Search-Direction Problem

Unbounded Search-Direction Problem

One issue that can come up when applying the feasible-directions method is that the linear optimization problem used to find a search direction can be unbounded.

This can obviously happen if the nonlinear optimization problem that we begin with is unbounded.

However, this can also happen on occasion even if the original problem is bounded.

Unbounded Search-Direction Problem

If the linear optimization problem used to find a search direction is unbounded, then we proceed by picking any direction in which the linear optimization problem is unbounded.

This is because the ultimate purpose of the search-direction problem is to find *any* feasible direction in which the objective function of the original nonlinear optimization problem improves.

Unbounded Search-Direction Problem

If the search-direction problem is unbounded, that tells us that there are directions in which the Taylor approximation of the original objective function improves without any limit.

Any such feasible direction of unboundedness will suffice for the feasible-directions method.

We then proceed to conduct a line search, but remove the restriction that $\alpha^k \leq 1$ (for that iteration only). We illustrate this with the following example.

Example

Example

Consider the linearly-constrained nonlinear optimization problem:

$$\begin{aligned} \min_x \; & f(x) = (x_1 - 3)^2 + (x_2 - 2)^2 \\ \text{s.t.} \; & -x_1 \leq 0 \\ & -x_2 \leq 0. \end{aligned}$$

We can easily confirm, using the KKT condition, that $x^* = (3, 2)^\top$ is a global optimum of this problem. Thus, this problem is not unbounded.

Example

Feasible

Let us now start from the point $x^0 = (0,0)^\top$ and attempt to conduct one iteration of the feasible-directions method. To do so, we first compute the gradient of the objective function, which is:

$$\nabla f(x) = \begin{pmatrix} 2(x_1 - 3) \\ 2(x_2 - 2) \end{pmatrix}.$$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Example

Thus, we find the search direction by solving the linear optimization problem:

$$\min_{d^0} \; f(x^0) + (d^0 - x^0)^\top \nabla f(x^0) = 13 - 8d_1^0 - 6d_2^0$$
$$\text{s.t.} \quad -d_1^0 \leq 0$$
$$-d_2^0 \leq 0.$$

$$\min_{d^k \in \mathbb{R}^n} \; f(x^k) + (d^k - x^k)^\top \nabla f(x^k)$$
$$\text{s.t.} \quad Ad^k - b \leq 0$$

Example

$$\begin{aligned}
\min_{d^0} \; & 13 - 8d_1^0 - 6d_2^0 \\
\text{s.t.} \quad & -d_1^0 \leq 0 \\
& -d_2^0 \leq 0.
\end{aligned}$$

Example

This problem is unbounded, because we can make either of $d_1^0 \to +\infty$ or $d_2^0 \to +\infty$, which is feasible and makes the objective function, $13 - 8d_1^0 - 6d_2^0$, arbitrarily small. In this case, we pick any direction in which the objective function is unbounded, and for simplicity, we take $d^0 = (1, 1)^\top$.

We next proceed to conducting an exact line search, which we do by solving the problem:

Example

$$\min_{\alpha^0} \; f(x^0 + \alpha^0 \cdot (d^0 - x^0)) = (\alpha^0 - 3)^2 + (\alpha^0 - 2)^2$$
$$\text{s.t.} \;\; 0 \leq \alpha^0.$$

Example

$$\min_{\alpha^0} \ (\alpha^0 - 3)^2 + (\alpha^0 - 2)^2$$
$$\text{s.t.} \quad 0 \leq \alpha^0.$$

Example

Note that, as discussed before, we remove the restriction that $\alpha \leq 1$. Solving this problem gives $\alpha^0 = 5/2$, meaning that $x^1 = (5/2, 5/2)^\top$. We can confirm that this point is feasible in the constraints.

Example

Moreover, we have:

$$f(x^0) = 13,$$

while:

$$f(x^1) = 1/2,$$

confirming that the objective-function value improves in going from x^0 to x^1. Note that because $x^1 \neq (3, 2)^\top$, the feasible directions algorithm does not terminate at this point. Indeed, it is easy to confirm that there is a new search direction that can be found by solving the new search-direction problem starting from the point x^1.

Feasible-Directions Algorithm for Linearly Constrained Nonlinear Optimization Problems

procedure FEASIBLE-DIRECTIONS ALGORITHM

  $k \leftarrow 0$ ▷ Set iteration counter to 0

  $\tau \leftarrow 0$

  Fix x^0 ▷ Fix a starting point

  if $Ax^0 - b \not\leq 0$ **then** ▷ If starting point is infeasible

    $x^0 \leftarrow \arg\min_x \{0^\top x | Ax - b \leq 0\}$ ▷ Find feasible starting point

  end if

  repeat

    $d^k \leftarrow \arg\min_x \{f(x^k) + (d^k - x^k)^\top \nabla f(x^k) | Ad^k - b \leq 0\}$ ▷ Find search direction

    if $(d^k - x^k)^\top \nabla f(x^k) = 0$ **then**

      $\tau \leftarrow 1$

    else

      if $\min_x \{f(x^k) + (d^k - x^k)^\top \nabla f(x^k) | Ad^k - b \leq 0\}$ is bounded **then**

        Determine step size, $\alpha^k \leq 1$

      else

        Determine step size, α^k

      end if

      $x^{k+1} \leftarrow x^k + \alpha^k \cdot (d^k - x^k)$ ▷ Update point

      $k \leftarrow k + 1$ ▷ Update iteration counter

    end if

  until $\tau = 1$ **or** other termination criteria met

end procedure

Feasible-Directions Algorithm for Linearly Constrained Nonlinear Optimization Problems

Lines 8 through 21 are the main iterative loop.

Line 9 solves the linear optimization problem that is used to determine the new search direction.

Lines 10 and 11 test the standard termination criterion. If $(d^k - x^k)^\top \nabla f(x^k) = 0$ then we set $\tau \leftarrow 1$, which will terminate the algorithm in Line 21.

Otherwise, we continue with the line search in Lines 13 through 17.

Feasible-Directions Algorithm for Linearly Constrained Nonlinear Optimization Problems

If the most recently solved search-direction problem is bounded, the step size is limited to be no greater than 1 (Line 14). Otherwise, the step size has no upper bound (Line 16). Note that any line search method can be employed here, so long as the appropriate bounds on α^k are imposed. Lines 18 and 19 update the point and iteration counter. As with the other iterative algorithms discussed in this chapter, other termination criteria can be employed in Line 21.

This is it!