Sensitivity

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What

Sensitivity

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The subject of **sensitivity analysis** is concerned with estimating how data changes to a nonlinear optimization problem affect the optimal objective-function value.

The following Sensitivity Property explains how this sensitivity analysis is conducted with nonlinear problems.

Sensitivity

Consider the problem:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \;& f(x) \\
\text{s.t.} \;& h_1(x) = 0 \\
& h_2(x) = 0 \\
& \vdots \\
& h_m(x) = 0 \\
& g_1(x) \leq 0 \\
& g_2(x) \leq 0 \\
& \vdots \\
& g_r(x) \leq 0.
\end{aligned}
$$

Suppose x^* is a local minimum and $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*, \mu_1^*, \mu_2^*, \ldots, \mu_r^*$ are Lagrange multipliers associated with the equality and inequality constraints.

Sensitivity

Consider the alternate equality- and inequality-constrained nonlinear optimization problem:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \;& f(x) \\
\text{s.t.} \;& h_1(x) = c_1 \\
& h_2(x) = c_2 \\
& \vdots \\
& h_m(x) = c_m \\
& g_1(x) \le d_1 \\
& g_2(x) \le d_2 \\
& \vdots \\
& g_r(x) \le d_r,
\end{aligned}
$$

and let $\hat{x}$ be a local minimum of this problem.

Sensitivity

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & f(x) \\ \text{s.t. } & h_1(x) = c_1 \\ & h_2(x) = c_2 \\ & \vdots \\ & h_m(x) = c_m \\ & g_1(x) \leq d_1 \\ & g_2(x) \leq d_2 \\ & \vdots \\ & g_r(x) \leq d_r \end{aligned}$$

So long as $|c_1|, |c_2|, \ldots, |c_m|, |d_1|, |d_2|, \ldots, |d_r|$ are sufficiently small, we can estimate the objective-function value of the new problem as:

$$f(\hat{x}) \approx f(x^*) - \sum_{i=1}^{m} \lambda_i^* c_i - \sum_{j=1}^{r} \mu_j^* d_j.$$

Sensitivity

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = c_1 \\ & h_2(x) = c_2 \\ & \vdots \\ & h_m(x) = c_m \\ & g_1(x) \le d_1 \\ & g_2(x) \le d_2 \\ & \vdots \\ & g_r(x) \le d_r \end{aligned}$$

Thus

$$\lambda_i^* = -\frac{\partial f(x)}{\partial c_i}\bigg]_{x=x^*}$$

$$\mu_i^* = -\frac{\partial f(x)}{\partial d_i}\bigg]_{x=x^*}$$

Sensitivity: Example

Sensitivity: Example

Consider the Packing-Box Problem:

$$
\begin{aligned}
\max_{h,w,u} \quad & hwu \\
\text{s.t.} \quad & 2wh + 2uh + 6wu \leq 60 \\
& h \geq 0 \\
& w \geq 0 \\
& u \geq 0
\end{aligned}
$$

Sensitivity: Example

In standard form this problem is:

$$\begin{aligned}
\min_{h,w,u} \ & f(h,w,u) = -hwu \\
\text{s.t.} \ & g_1(h,w,u) = 2wh + 2uh + 6wu - 60 \leq 0 && (\mu_1) \\
& g_2(h,w,u) = -h \leq 0 && (\mu_2) \\
& g_3(h,w,u) = -w \leq 0 && (\mu_3) \\
& g_3(h,w,u) = -u \leq 0, && (\mu_4)
\end{aligned}$$

Sensitivity: Example

where the Lagrange multiplier associated with each constraint is indicated in the parentheses to right of it.

Sensitivity: Example

We know that:

$$(h, w, u, \mu_1, \mu_2, \mu_3, \mu_4) \approx (4.24, 1.41, 1.41, 0.35, 0, 0, 0),$$

is the unique solution to the KKT conditions and is a local and global minimum of the problem.

Sensitivity: Example

We wish to know the effect of increasing the amount of cardboard available to 62 cm^2 and requiring the box to be at least 0.4 cm wide. In other words, we would like to estimate the optimal objective-function value of the following problem:

$$\max_{h,w,d} \; hwu$$

$$\text{s.t.} \quad 2wh + 2uh + 6wu \leq 62$$

$$h \geq 0$$

$$w \geq 0.4$$

$$u \geq 0.$$

Sensitivity: Example

To apply the Sensitivity Property to answer this question, we must convert the constraints of this problem to have the same left-hand sides as the standard form problem.

This is because the Sensitivity Property only tells us how to estimate the effect of changes to the right-hand side of constraints.

Sensitivity: Example

We can write the problem with additional cardboard and the minimum width requirement as:

$$
\begin{aligned}
\min_{h,w,u} \ & f(h,w,u) = -hwu \\
\text{s.t.} \ & g_1(h,w,u) = 2wh + 2uh + 6wu - 60 \le 2 \\
& g_2(h,w,u) = -h \le 0 \\
& g_3(h,w,u) = -w \le -0.4 \\
& g_3(h,w,u) = -u \le 0.
\end{aligned}
$$

$$(h, w, u, \mu_1, \mu_2, \mu_3, \mu_4) \approx (4.24, 1.41, 1.41, 0.35, 0, 0, 0)$$

Sensitivity: Example

Applying the Sensitivity Property, we can estimate the new optimal objective-function value as:

$$f(\hat{x}) \approx \boxed{f(x^*)} - 2\mu_1^* - 0\mu_2^* + 0.4\mu_3^* - 0\mu_4^* = \boxed{-8.43} - 0.7 = -9.13.$$

Sensitivity: Example

The Sensitivity Property shows the objective function decreasing when we increase the amount of available cardboard.

Recall, however, that the original problem is a maximization. We change the objective to a minimization by multiplying the objective through by -1 in order to apply the KKT condition.

Thus, the volume of the box *increases* by approximately 0.7 cm^3 when we add the cardboard and impose the minimum-width requirement.

Sensitivity: Proof

Sensitivity: Proof

$$\left.\begin{array}{lll}(\text{P1}) & \min_x & f(x) \\ & \text{s.t.} & h(x) = 0 : \lambda\end{array}\right\} \Rightarrow x^*, \lambda^*$$

$$\left.\begin{array}{lll}(\text{P2}) & \min_x & f(x) \\ & \text{s.t.} & h(x) = c : \lambda\end{array}\right\} \Rightarrow x(c), \lambda(c)$$

$$\left.\right\} \quad \boxed{\begin{array}{l} x(c)]_{c=0} = x^* \\ \lambda(c)]_{c=0} = \lambda^* \end{array}}$$

$||c||$ small enough around 0

$f(x)$ convex over the convex set $h(x) = 0$

$f(x) : \mathbb{R}^n \to \mathbb{R}$ and $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ differentiable.

Sensitivity: Proof

Gradient of $h(x(c))$:

$$\nabla_c h(x(c))]_{c=0} = \nabla_x h(x^*) \nabla_c x(0) \qquad (1)$$

On the other hand:

$$h_1(x(c)) = c_1$$
$$\vdots$$
$$h_1(x(c)) = c_m$$

Then: $\nabla_c h(x(c)) = I = \nabla_c h(x(c))]_{c=0}$

Considering (1), we get:

$$\nabla_x h(x^*) \nabla_c x(0) = I \qquad (2)$$

Sensitivity: Proof

KKT conditions for (P1):

$$(\nabla_x f(x^*))^\top + \lambda^{*\top} \nabla_x h(x^*) = 0$$

Multiplying by $\nabla_c x(0)$:

$$(\nabla_x f(x^*))^\top \nabla_c x(0) + \lambda^{*\top} \nabla_x h(x^*) \nabla_c x(0) = 0$$

Considering (2) ($\nabla_x h(x^*) \nabla_c x(0) = I$):

$$(\nabla_x f(x^*))^\top \nabla_c x(0) + \lambda^{*\top} I = 0$$

Thus:

$$(\nabla_x f(x^*))^\top \nabla_c x(0) = -\lambda^{*\top} \quad (3)$$

Sensitivity: Proof

Gradient of $f(x(c))$:

$$\nabla_c f(x(c))^\top\Big]_{c=0} = (\nabla_x f(x^*))^\top \nabla_c x(0)$$

Considering (3) ($(\nabla_x f(x^*))^\top \nabla_c x(0) = -\lambda^{*\top}$):

$$\nabla_c f(x(c))\Big]_{c=0} = -\lambda^*$$

And:

$$\frac{\partial f(x(c))}{\partial c_i}\Bigg]_{c=0} = \boxed{\frac{\partial f(x^*)}{\partial c_i} = -\lambda_i^*}$$

This is it!