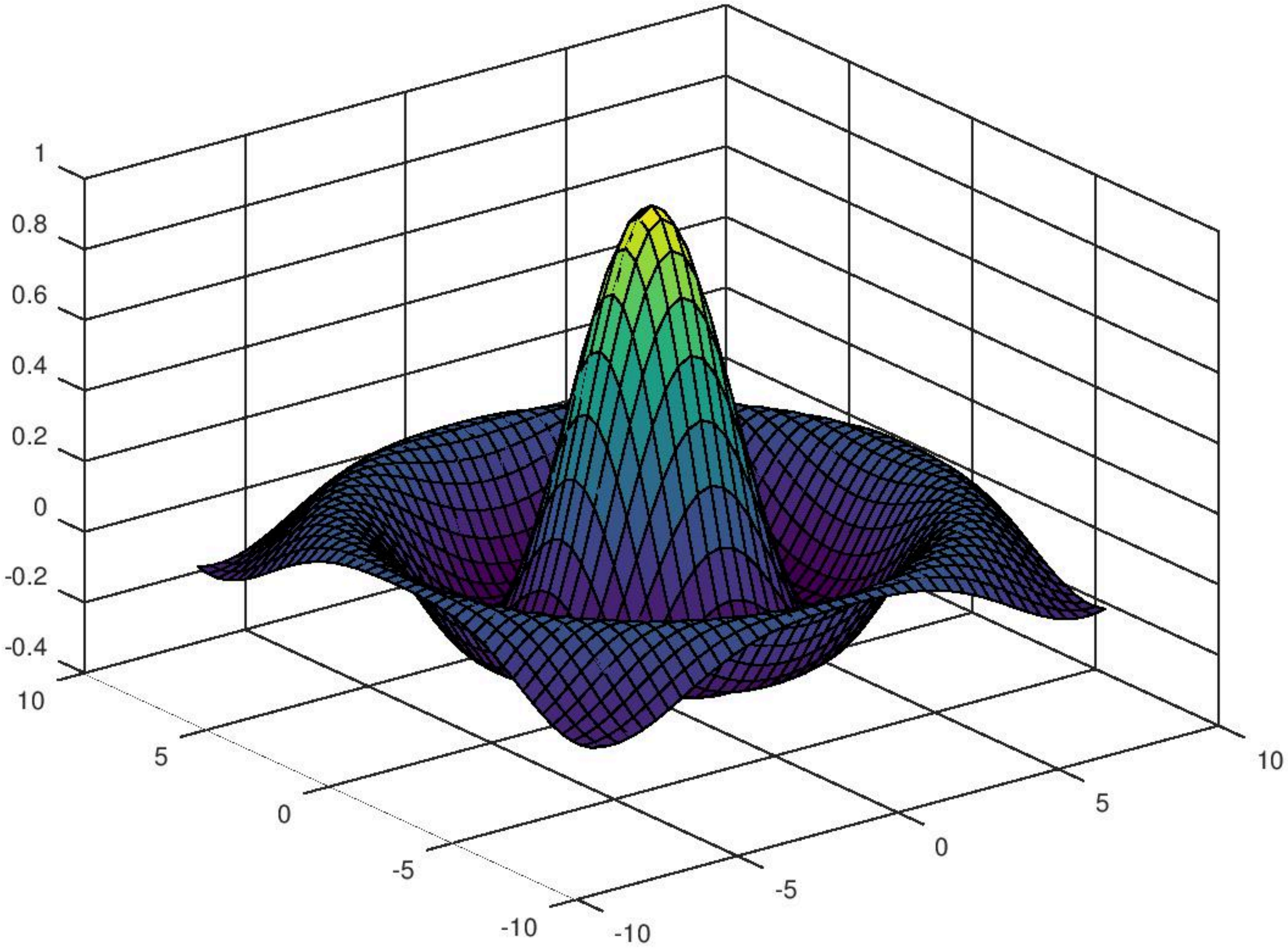
E&ICP: Optimality Conditions



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What

Equality- & Inequality-Constrained Nonlinear Optimization Problems

1. First-Order Necessary Condition
2. Regularity
3. Second-Order Sufficient Condition

FONC

Consider an inequality-constrained nonlinear optimization problem of the form:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = 0 \\
& h_2(x) = 0 \\
& \vdots \\
& h_m(x) = 0 \\
& g_1(x) \leq 0 \\
& g_2(x) \leq 0 \\
& \vdots \\
& g_r(x) \leq 0
\end{aligned}
$$

FONC

If x^* is a local minimum of this problem, then there exist $(m+r)$ **Lagrange multipliers**, $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ and $\mu_1^*, \mu_2^*, \ldots, \mu_r^*$, such that:

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0$$

$$h_1(x^*) = 0$$

$$h_2(x^*) = 0$$

$$\vdots$$

$$h_m(x^*) = 0$$

$$0 \leq \mu_1^* \perp g_1(x^*) \leq 0$$

$$0 \leq \mu_2^* \perp g_2(x^*) \leq 0$$

$$\vdots$$

$$0 \leq \mu_r^* \perp g_r(x^*) \leq 0.$$

FONC

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0$$

$$h_1(x^*) = 0$$

$$h_2(x^*) = 0$$

$$\vdots$$

$$h_m(x^*) = 0$$

$$0 \le \mu_1^* \perp g_1(x^*) \le 0$$

$$0 \le \mu_2^* \perp g_2(x^*) \le 0$$

$$\vdots$$

$$0 \le \mu_r^* \perp g_r(x^*) \le 0.$$

Regularity

Regularity

The FONC for equality- and inequality-constrained problems has one additional technical requirement, which is known as regularity.

A point is said to be **regular** if the gradients of the binding constraint functions at that point are all linearly independent.

SOSC

SOSC

$$\nabla_x^2 f(x) + \lambda^T \nabla_x^2 h(x) + \mu^T \nabla_x^2 g(x) > 0$$

on the subspace

$$\{y : \nabla_x h(x) y = 0; \nabla_x g_j(x) y = 0, \forall j \in \mathcal{J}\}, \mathcal{J} = j : g_j(x) = 0, \mu_j > 0$$

SOSC

Note that

$$\{y : \nabla_x h(x)y = 0; \nabla_x g_j(x)y = 0, \forall j \in \mathcal{J}\}, \mathcal{J} = j : g_j(x) = 0, \mu_j > 0$$

is the tangent hyperplane at x, and that

$$\lambda^T \nabla_x^2 h(x) = \sum_{i=1}^{m} \lambda_i \nabla_x^2 h_i(x)$$

$$\mu^T \nabla_x^2 g(x) = \sum_{i=1}^{r} \mu_i \nabla_x^2 g_i(x)$$

This is it!