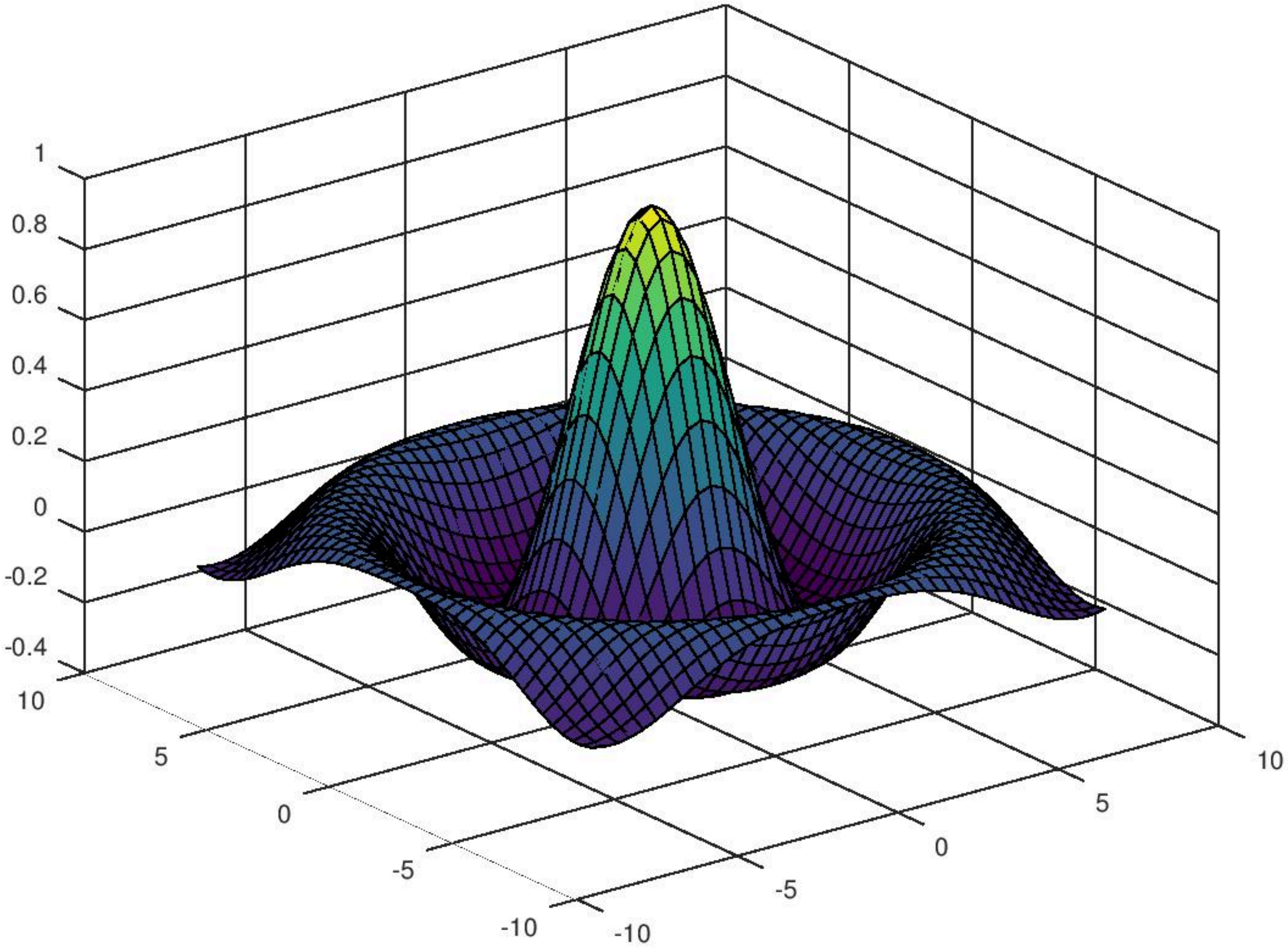
ICP: Optimality Conditions



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What

Inequality-Constrained Nonlinear Optimization Problems

1. First-Order Necessary Condition
2. Examples
3. Regularity
4. Second-Order Sufficient Condition
5. Examples

FONC

FONC

Consider an inequality-constrained nonlinear optimization problem of the form:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \;& f(x) \\
\text{s.t.} \;& g_1(x) \leq 0 \\
& g_2(x) \leq 0 \\
& \vdots \\
& g_r(x) \leq 0.
\end{aligned}
$$

$$f(x) : \mathbb{R}^n \to \mathbb{R}$$

$$g_i(x) : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, r$$

If x^* is a local minimum of this problem, then there exist r **Lagrange multipliers**, $\mu_1^*, \mu_2^*, \dots, \mu_r^*$, such that:

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$$\nabla f(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0$$

$$\mu_1^* \geq 0$$

$$\mu_2^* \geq 0$$

$$\vdots$$

$$\mu_r^* \geq 0$$

$$\mu_1^* g_1(x^*) = 0$$

$$\mu_2^* g_2(x^*) = 0$$

$$\vdots$$

$$\mu_r^* g_r(x^*) = 0.$$

FONC

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t.} \quad g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

$$\vdots$$

$$g_r(x) \leq 0$$

$$f(x) : \mathbb{R}^n \to \mathbb{R}$$

$$g_i(x) : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, r$$

$$\nabla f(x^*) + \sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0$$

$$0 \leq \mu_1^* \perp g_1(x^*) \leq 0$$

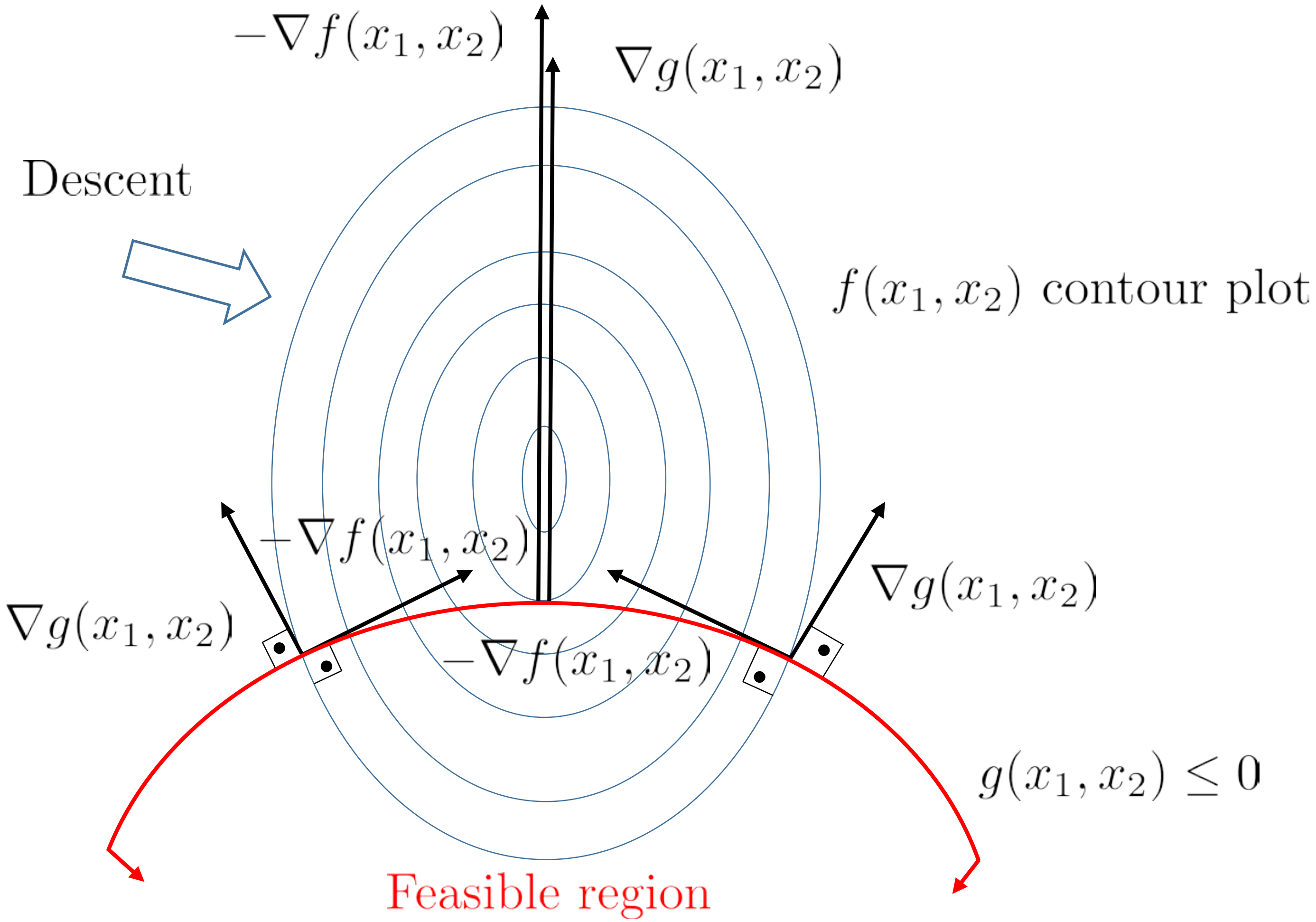
$$0 \leq \mu_2^* \perp g_2(x^*) \leq 0$$

$$\vdots$$

$$0 \leq \mu_r^* \perp g_r(x^*) \leq 0$$

FONC

$$\min_{x_1,x_2} \; f(x_1,x_2)$$
$$\text{s.t.} \;\; g(x_1,x_2) \le 0$$



$$\nabla f(x_1,x_2) + \mu \nabla g(x_1,x_2) = 0, \qquad \mu > 0$$

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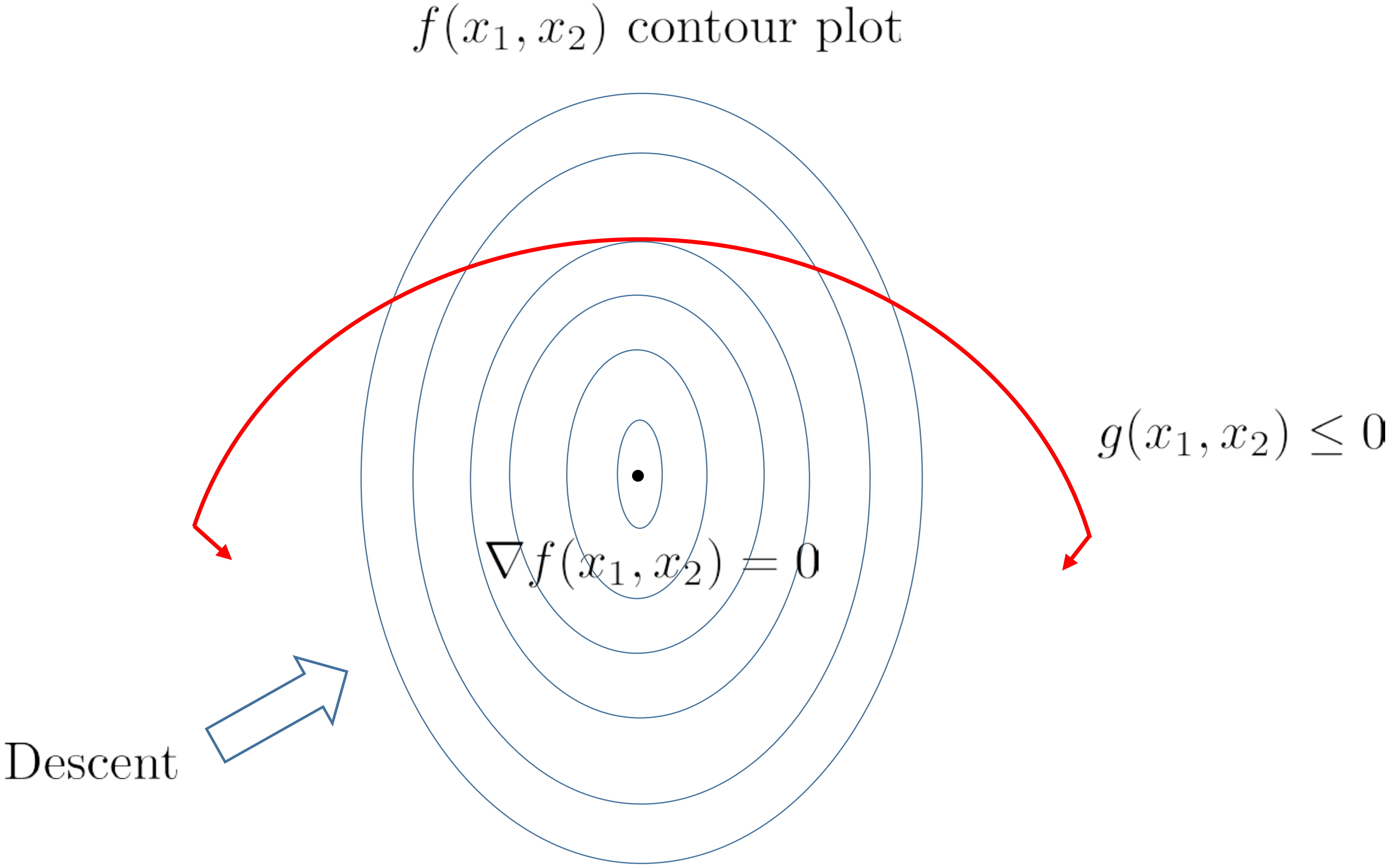
$$\min_{x_1, x_2} \; f(x_1, x_2)$$
$$\text{s.t.} \;\; g(x_1, x_2) \leq 0$$



$$\nabla f(x_1, x_2) + \mu \nabla g(x_1, x_2) = 0, \qquad \mu = 0$$

Regularity

Regularity

The FONC for inequality-constrained problems has one additional technical requirement, which is known as regularity.

A point is said to be **regular** if the gradients of the binding constraint functions at that point are all linearly independent.

FONC: Example

FONC: Example

$$
\begin{aligned}
\min_{x} \quad & f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2 \\
\text{s.t.} \quad & g_1(x) = x_1^2 + x_2^2 - 9 \leq 0 \\
& g_2(x) = -x_1 + 2x_2 + 1 \leq 0.
\end{aligned}
$$

FONC: Example

$$\begin{aligned} \min_{x} \ & f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2 \\ \text{s.t.} \ & g_1(x) = x_1^2 + x_2^2 - 9 \leq 0 \\ & g_2(x) = -x_1 + 2x_2 + 1 \leq 0. \end{aligned}$$

To write out the KKT conditions we define two Lagrange multipliers, μ_1 and μ_2, associated with the two inequality constraints.

The KKT conditions and the constraints of the original problem are then:

FONC: Example

$$\min_{x} \; f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2$$
$$\text{s.t.} \; g_1(x) = x_1^2 + x_2^2 - 9 \leq 0$$
$$g_2(x) = -x_1 + 2x_2 + 1 \leq 0$$

$$4x_1 + 2x_2 + 1 + 2\mu_1 x_1 - \mu_2 = 0$$
$$2x_1 + 4x_2 + 1 + 2\mu_1 x_2 + 2\mu_2 = 0$$
$$0 \leq \mu_1 \perp x_1^2 + x_2^2 - 9 \leq 0$$
$$0 \leq \mu_2 \perp -x_1 + 2x_2 + 1 \leq 0$$

FONC: Example

$$\min_x \; f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2$$
$$\text{s.t.} \; g_1(x) = x_1^2 + x_2^2 - 9 \leq 0$$
$$g_2(x) = -x_1 + 2x_2 + 1 \leq 0$$

Note that these conditions are considerably more difficult to work with than the FONC in the unconstrained and equality-constrained cases.

This is because we now have a system of equations and inequalities, the latter coming from the inequality constraints and the non-negativity restrictions on the Lagrange multipliers associated with them.

FONC: Example

As such, we approach equality- and inequality-constrained problems by conjecturing which of the inequality constraints are binding and non-binding, and then solving the resulting the KKT conditions.

We must examine all combinations of binding and non-binding constraints until we find all solutions to the KKT conditions.

No constraint binding

FONC: Example

With the problem at hand, let us first consider the case in which none of the inequality constraints are binding.

The complementary-slackness conditions then imply that $\mu_1 = 0$ and $\mu_2 = 0$. The gradient conditions are then simplified to:

$$4x_1 + 2x_2 + 1 = 0$$

$$2x_1 + 4x_2 + 1 = 0.$$

No constraint binding

FONC: Example

Solving the two equations gives:

$$(x_1, x_2) = (-1/6, -1/6),$$

meaning we have found as a possible KKT point:

$$(x_1, x_2, \mu_1, \mu_2) = (-1/6, -1/6, 0, 0).$$

No constraint binding

FONC: Example

However, we only found these values of x and μ by assuming which of the constraints are binding and non-binding (to determine the values of the μ's) and then solving for the x's in the gradient conditions.

We must still check to ensure that these values satisfy all of the other conditions.

If we do so, we see that the second inequality constraint is violated, meaning that this is not a solution to the KKT conditions.

1st binding, 2nd non-binding

FONC: Example

We next consider the case in which the first inequality constraint is binding and the second is non-binding.

The complementary-slackness conditions then imply that $\mu_2 = 0$ whereas we cannot make any determination about μ_1. Thus, the gradient conditions become:

$$4x_1 + 2x_2 + 1 + 2\mu_1 x_1 = 0$$

$$2x_1 + 4x_2 + 1 + 2\mu_1 x_2 = 0.$$

1st binding, 2nd non-binding

FONC: Example

This is a system of two equations with three unknowns. We, however, have one additional equality that the x's must satisfy, which is the first inequality constraint.

Because we are assuming in this case that this constraint is binding, we impose it as a third equation:

$$x_1^2 + x_2^2 - 9 = 0.$$

1st binding, 2nd non-binding

FONC: Example

$$4x_1 + 2x_2 + 1 + 2\mu_1 x_1 = 0$$

$$2x_1 + 4x_2 + 1 + 2\mu_1 x_2 = 0$$

$$x_1^2 + x_2^2 - 9 = 0$$

1st binding, 2nd non-binding

FONC: Example

Solving this system of equations gives:

$$(x_1, x_2, \mu_1) \approx (-2.12, -2.12, -2.76),$$

$$(x_1, x_2, \mu_1) \approx (1.86, -2.36, -1.00),$$

and:

$$(x_1, x_2, \mu_1) \approx (-2.36, 1.86, -1.00),$$

1st binding, 2nd non-binding

FONC: Example

meaning that:

$$(x_1, x_2, \mu_1, \mu_2) \approx (-2.12, -2.12, -2.76, 0),$$

$$(x_1, x_2, \mu_1, \mu_2) \approx (1.86, -2.36, -1.00, 0),$$

and:

$$(x_1, x_2, \mu_1, \mu_2) \approx (-2.36, 1.86, -1.00, 0),$$

are candidate KKT points. However, because μ_1 is negative in all three of these vectors, these are not KKT points.

1st non-binding, 2nd binding

FONC: Example

The third case that we examine is the one in which the first inequality constraint is non-binding and the second inequality is binding.

The complementary-slackness conditions imply that $\mu_1 = 0$ whereas we cannot make any determination regarding the value of μ_2.

1st non-binding, 2nd binding

FONC: Example

Thus, the simplified gradient conditions and the second inequality constraint (which we impose as an equality) are:

$$4x_1 + 2x_2 + 1 - \mu_2 = 0$$

$$2x_1 + 4x_2 + 1 + 2\mu_2 = 0$$

$$-x_1 + 2x_2 + 1 = 0.$$

1st non-binding, 2nd binding

FONC: Example

Solving these three equations gives:

$$(x_1, x_2, \mu_2) = (1/14, -13/28, 5/14),$$

meaning that we have found:

$$(x_1, x_2, \mu_1, \mu_2) = (1/14, -13/28, 0, 5/14),$$

as a possible solution KKT point. Moreover, when we check the remaining conditions, we find that they are all satisfied, meaning that this is indeed a KKT point.

Both constraints binding

FONC: Example

The last possible case that we examine is the one in which both of the inequality constraints are binding. In this case complementary slackness does not allow us to fix any of the μ's equal to zero.

Both constraints binding

FONC: Example

Thus, we solve the following system of equations:

$$4x_1 + 2x_2 + 1 + 2\mu_1 x_1 - \mu_2 = 0$$

$$2x_1 + 4x_2 + 1 + 2\mu_1 x_2 + 2\mu_2 = 0$$

$$x_1^2 + x_2^2 - 9 = 0$$

$$-x_1 + 2x_2 + 1 = 0$$

FONC: Example

Both constraints binding

which has the solutions:

$$(x_1, x_2, \mu_1, \mu_2) \approx (-2.45, -1.73, -2.66, 0.81),$$

and:

$$(x_1, x_2, \mu_1, \mu_2) \approx (2.85, 0.93, -2.94, -2.49).$$

Clearly neither of these are KKT points, because both of them have negative values for μ_1.

FONC: Example

Thus, the only solution to the KKT conditions and the only candidate point that could be a local minimum is $(x_1^*, x_2^*) = (1/14, -13/28)$.

FONC

FONC

Algorithm for Finding KKT Points

```
procedure KKT Find
    Fix which inequalities are binding and non-binding
    Fix μ's for non-binding inequalities to zero
    Solve system of equations given by gradient conditions and binding inequalities (written as equalities)
    Check that x's satisfy constraints assumed to be non-binding and μ's are non-negative
end procedure
```

FONC: Example

$$\min_{x} \; f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 + x_2$$
$$\text{s.t.} \quad g_1(x) = x_1^2 + x_2^2 - 9 \leq 0$$
$$g_2(x) = -x_1 + 2x_2 + 1 \leq 0$$

Note that the Hessian of the objective function is:

$$\nabla^2 f(x) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix},$$

which is positive definite, meaning that the objective is a convex function.

FONC: Example

Moreover, the second inequality constraint is linear, which we know defines a convex feasible region. The Hessian of the first inequality constraint is:

$$\nabla^2 g(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is also positive definite, meaning that this constraint function is also convex.

FONC: Example

Thus, this problem is convex and any KKT point we find it guaranteed to be a global minimum.

This means that once we find the KKT point $(x_1, x_2, \mu_1, \mu_2) = (1/14, -13/28, 0, 5/14)$, we can stop and ignore the fourth case, because we have a global minimum.

SOSC

SOSC

$$\nabla_x^2 f(x) + \mu^T \nabla_x^2 g(x) > 0$$

on the subspace

$$\{y : \nabla_x g_j(x) y = 0, \forall j \in \mathcal{J}\}$$

$$\mathcal{J} = j : g_j(x) = 0, \mu_j > 0$$

SOSC

Note that

$$\{y : \nabla_x g_j(x) y = 0, \forall j \in \mathcal{J}\}, \mathcal{J} = j : g_j(x) = 0, \mu_j > 0$$

is the tangent hyperplane at x, and that

$$\mu^T \nabla_x^2 g(x) = \sum_{i=1}^{r} \mu_i \nabla_x^2 g_i(x)$$

This is it!