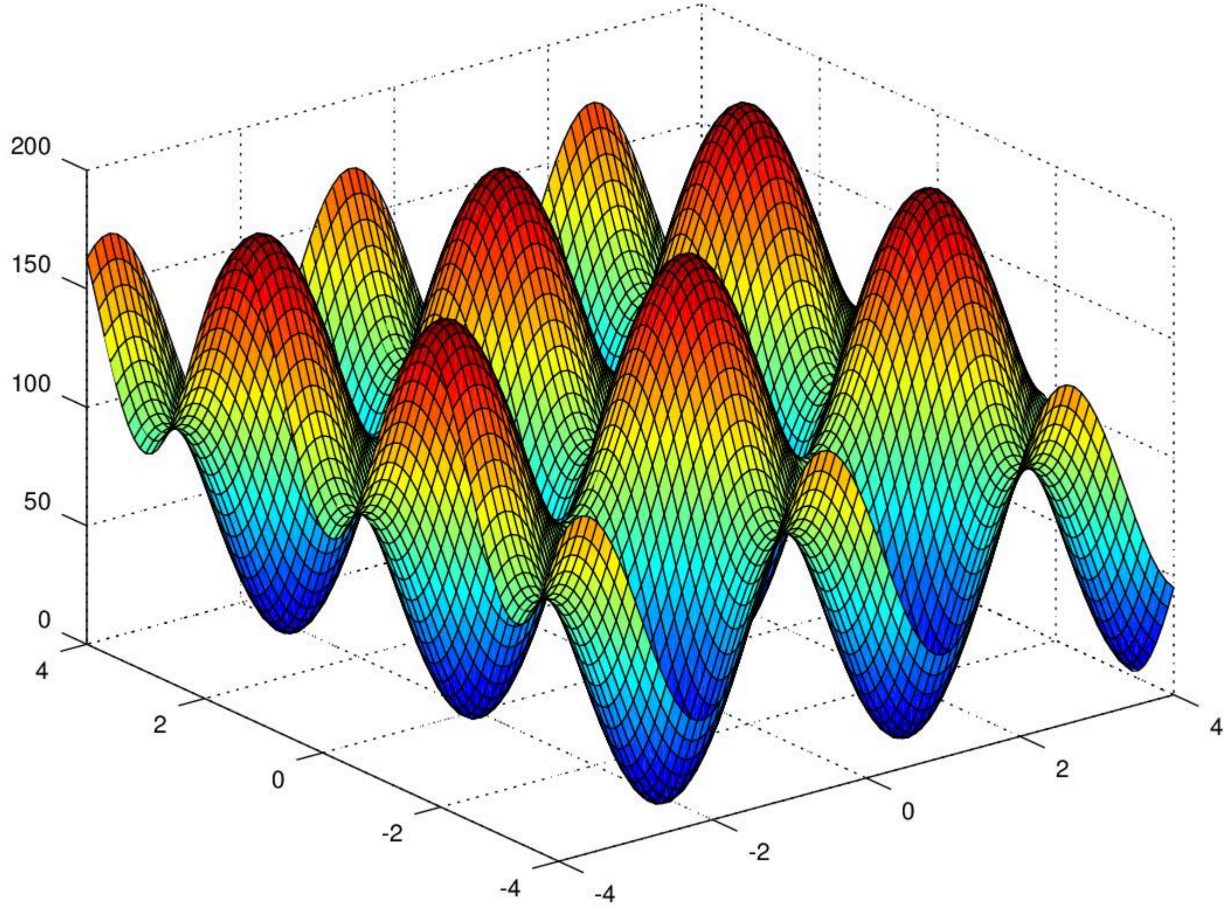
Introduction



A. J. Conejo, R. Sioshansi, 2016

THE OHIO STATE UNIVERSITY

What

1. Optimization problems
2. Convex sets
3. Convex functions
4. Global and local minima

Optimization Problem

Optimization Problem

An unconstrained nonlinear optimization problem has the general form:

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f(x)$ is the objective being minimized and there are no constraints on the decision variables.

Optimization Problem

$$\min_{x \in \mathbb{R}^n} \; f(x)$$

$$f(x) : \mathbb{R}^n \to \mathbb{R}$$

Optimization Problem

An equality-constrained nonlinear optimization problem has an objective function that is being minimized and a set of m equality constraints that have zeros on their right-hand sides.

The generic version of the equality-constrained nonlinear optimization problem is:

Optimization Problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ \text{s.t.} \; & h_1(x) = 0 \\ & h_2(x) = 0 \\ & \vdots \\ & h_m(x) = 0, \end{aligned}$$

where $f(x)$ is the objective function and $h_1(x), h_2(x), \ldots, h_m(x)$ are the m equality constraint functions.

Optimization Problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = 0 \\
& h_2(x) = 0 \\
& \vdots \\
& h_m(x) = 0
\end{aligned}
$$

$$f(x) : \mathbb{R}^n \to \mathbb{R}$$

$$h_i(x) : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$$

Optimization Problem

An equality- and inequality-constrained nonlinear optimization problem has an objective function that is being minimized, a set of m equality constraints that have zeros on their right-hand sides, and a set of r less-than-or-equal-to constraints that have zeros on their right-hand sides.

The generic version of the equality- and inequality-constrained problem is:

Optimization Problem

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = 0 \\
& h_2(x) = 0 \\
& \vdots \\
& h_m(x) = 0 \\
& g_1(x) \leq 0 \\
& g_2(x) \leq 0 \\
& \vdots \\
& g_r(x) \leq 0
\end{aligned}$$

$$f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$h_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \ldots, m$$

$$g_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \ldots, r$$

Optimization Problem

where $f(x)$ is the objective function,

$h_1(x), h_2(x), \ldots, h_m(x)$ are the m equality constraint functions, and

$g_1(x), g_2(x), \ldots, g_r(x)$ are the r inequality constraint functions.

Note that one could have a problem with only inequality constraints, i.e., $m = 0$, meaning that there are no equality constraints.

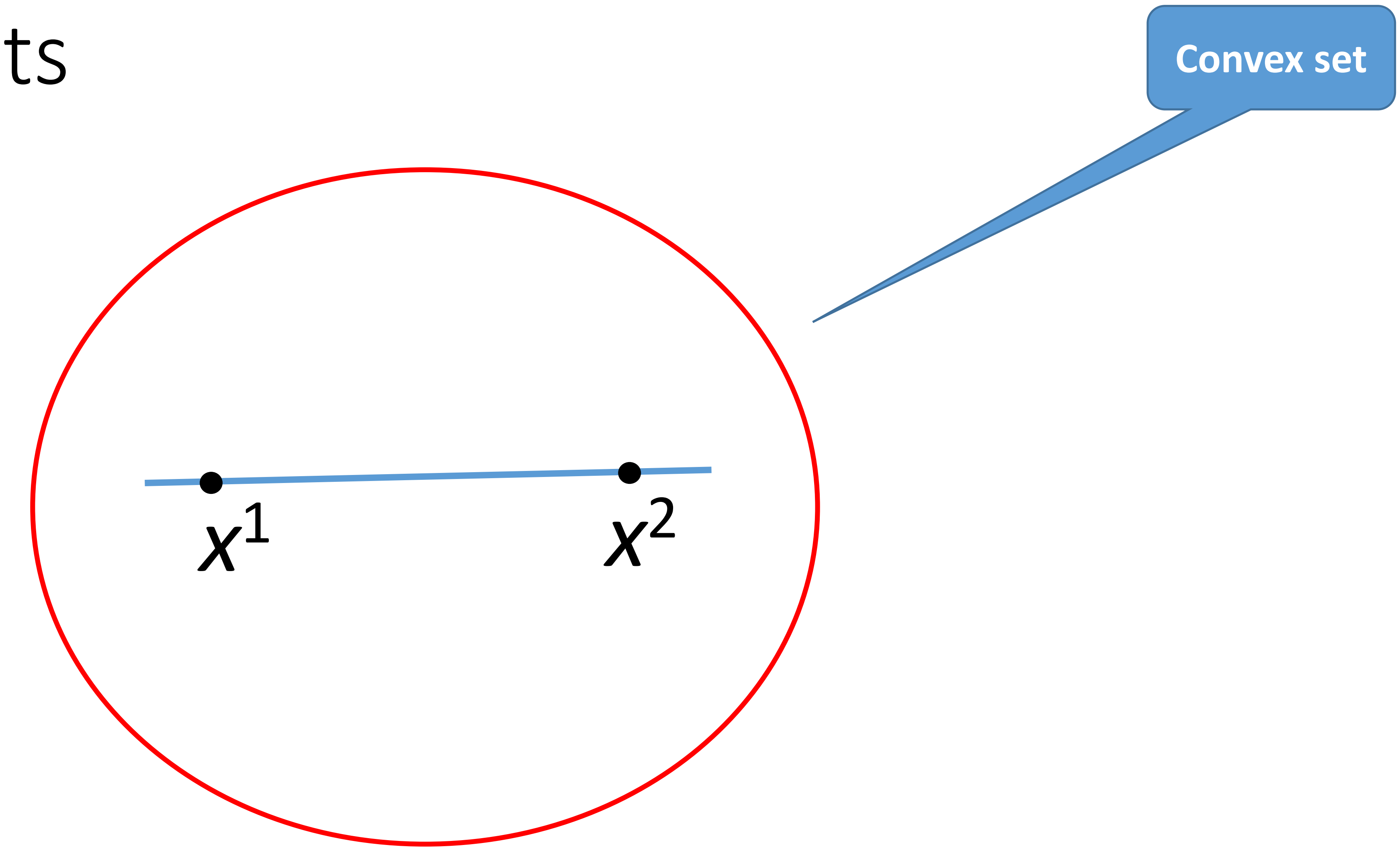
Convex Sets

Convex Sets

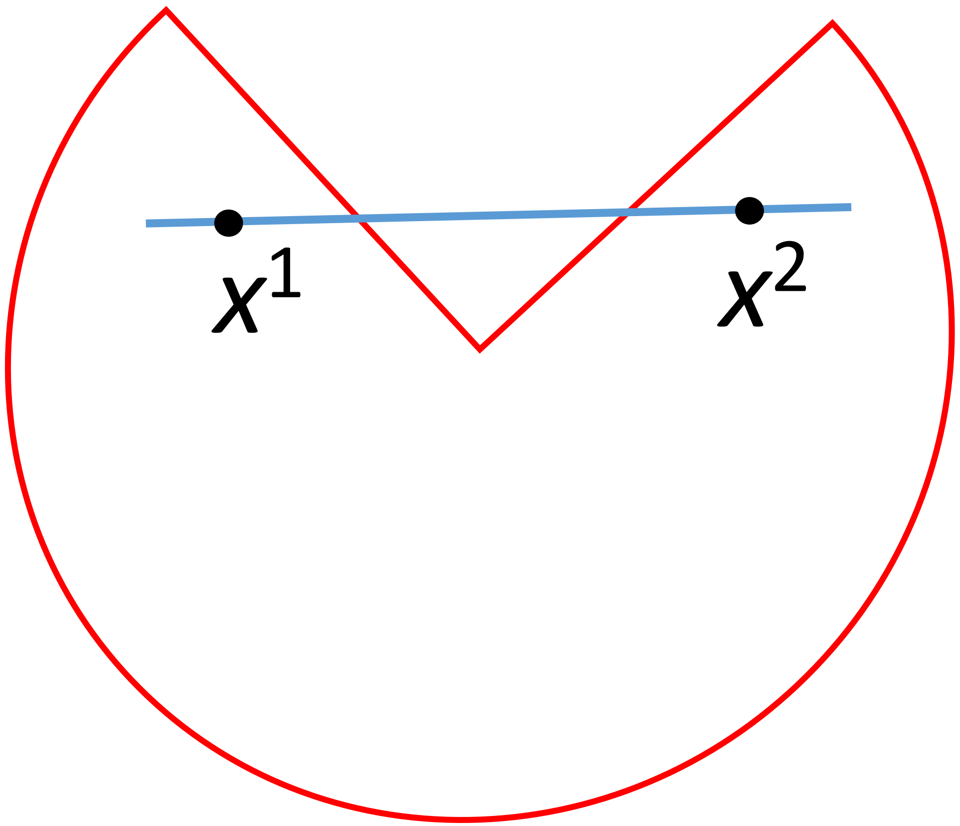
A set $X \subseteq \mathbb{R}^n$ is said to be a **convex set** if for any two points x^1 and $x^2 \in X$ and for any value of $\alpha \in [0, 1]$ we have that:

$$\alpha x^1 + (1 - \alpha)x^2 \in X.$$

Convex Sets



Convex Sets



Nonconvex set

Convex Functions

Convex Functions

Given a convex set, $X \subseteq \mathbb{R}^n$, a function defined on X is said to be a **convex function** on X if for any two points, x^1 and $x^2 \in X$, and for any value of $\alpha \in [0, 1]$ we have that:

$$\alpha f(x^1) + (1 - \alpha) f(x^2) \geq f(\alpha x^1 + (1 - \alpha) x^2).$$

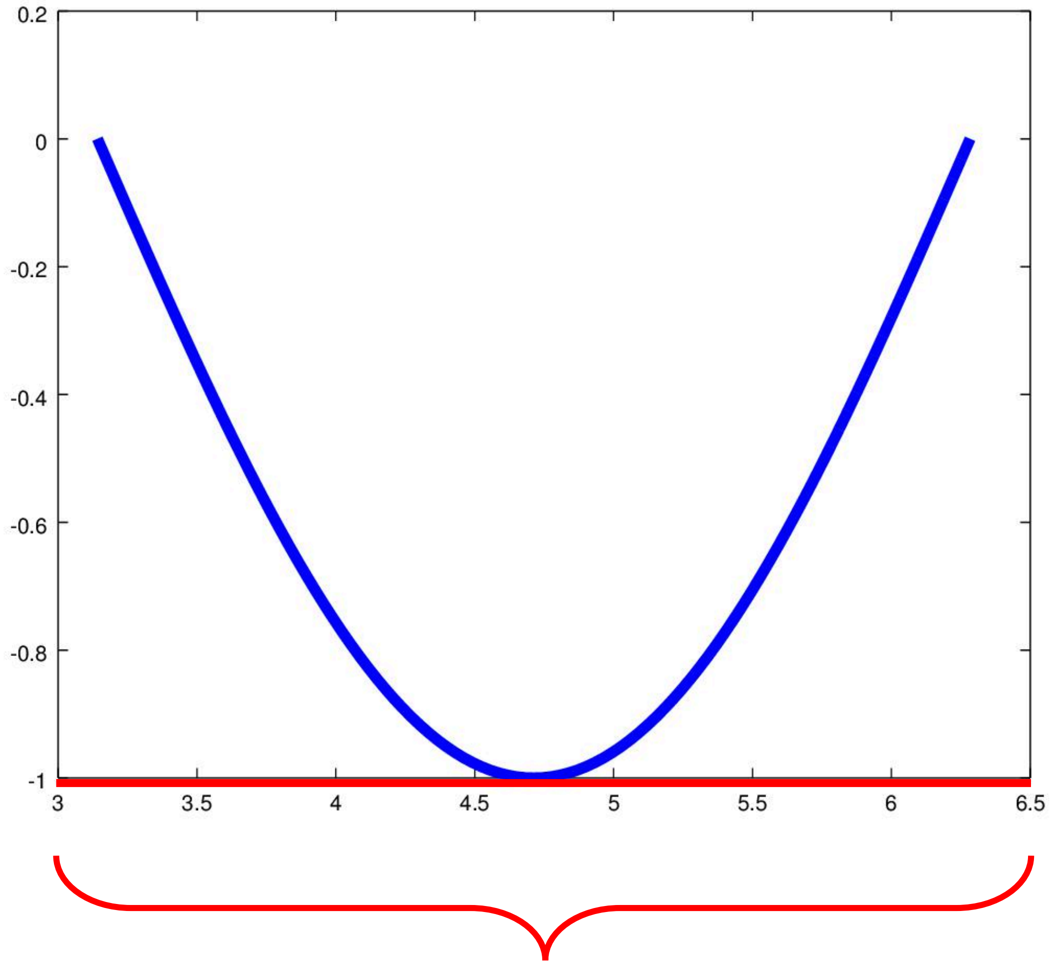
Convex Functions

$$\alpha f(x^1) + (1-\alpha) f(x^2) \geq f(\alpha x^1 + (1-\alpha) x^2)$$

Convex Functions

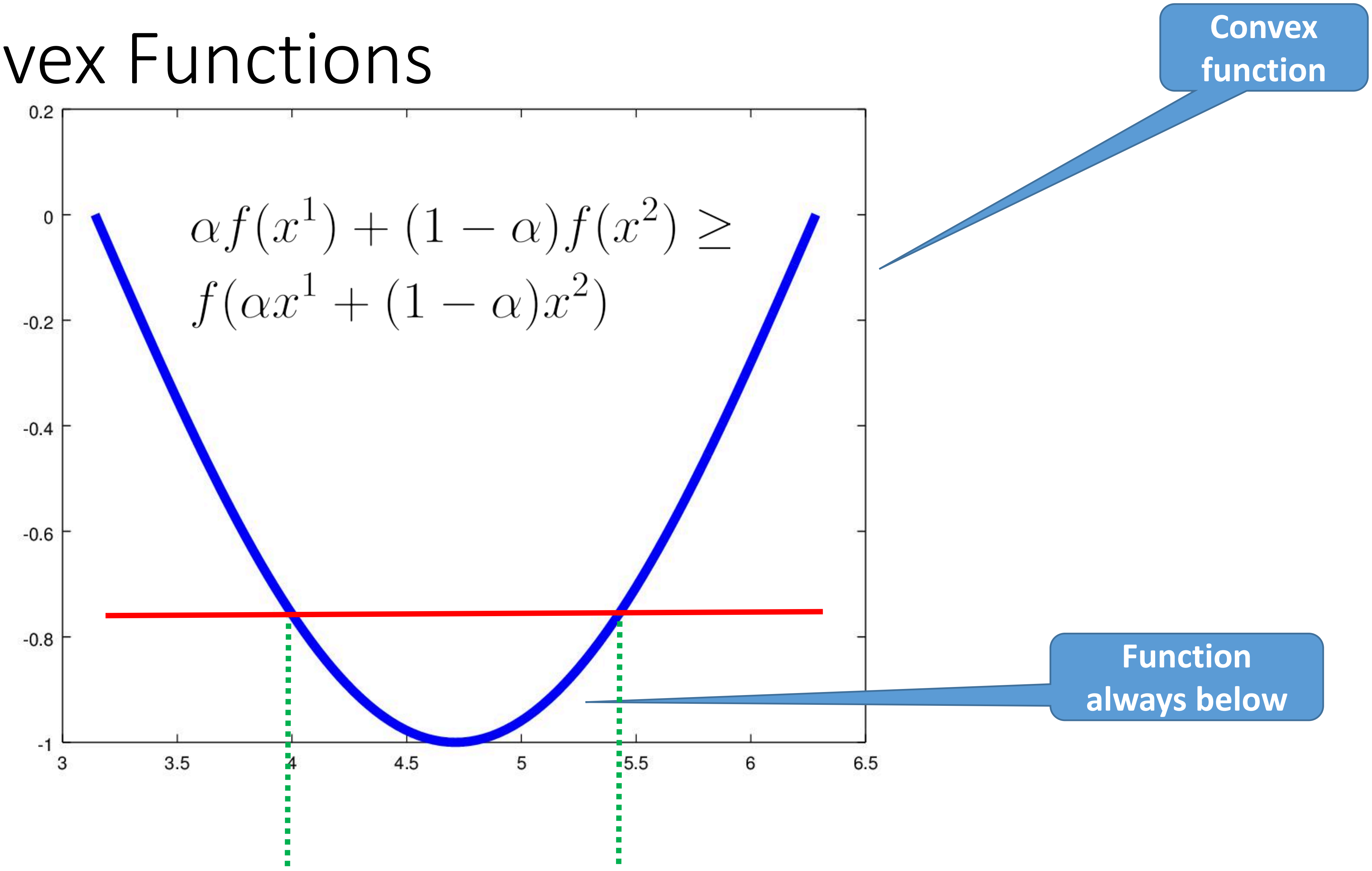
```
x = 1*pi:pi/100:2*pi;
y = sin(x);
plot(x,y,'LineWidth',5)
```

Convex Functions



Convex Set

Convex Functions

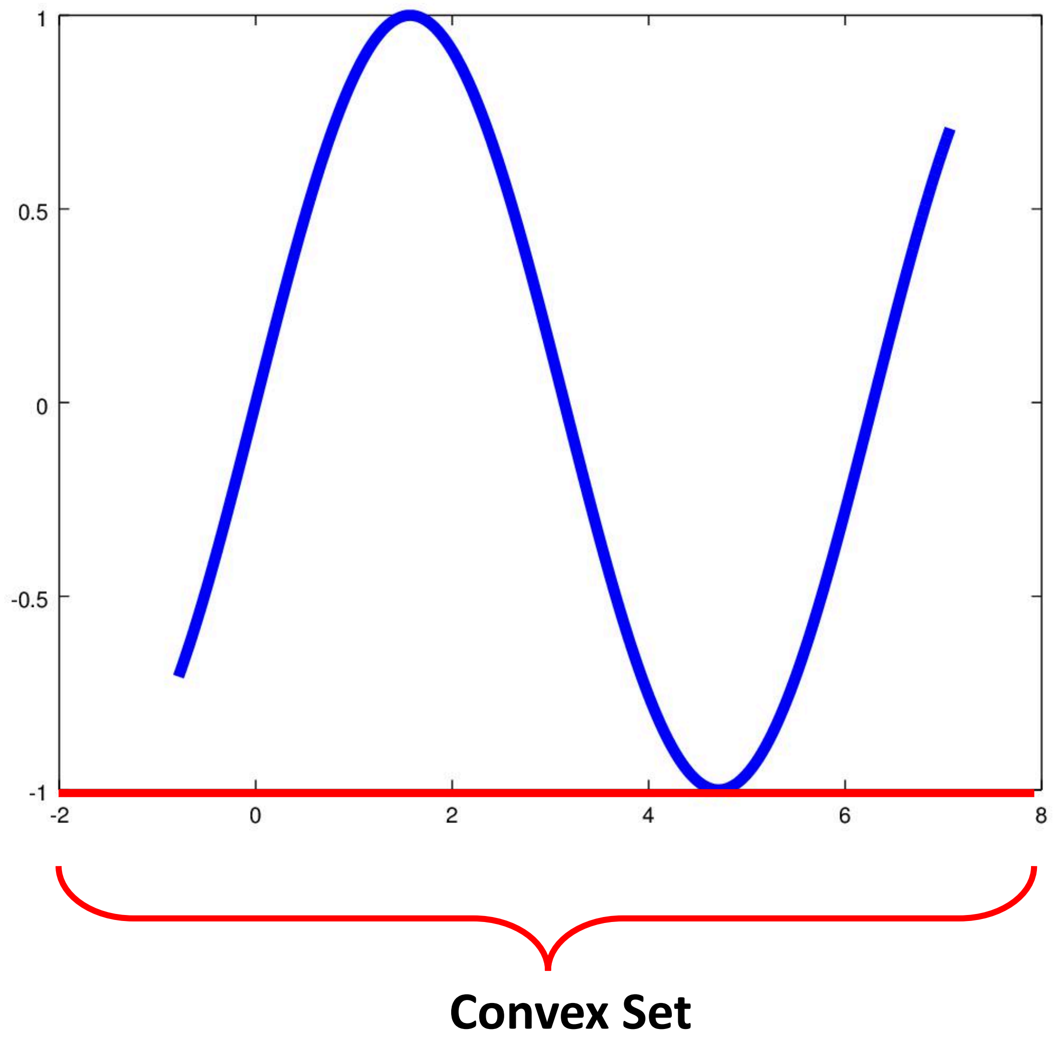


$$\alpha f(x^1) + (1-\alpha) f(x^2) \geq f(\alpha x^1 + (1-\alpha) x^2)$$

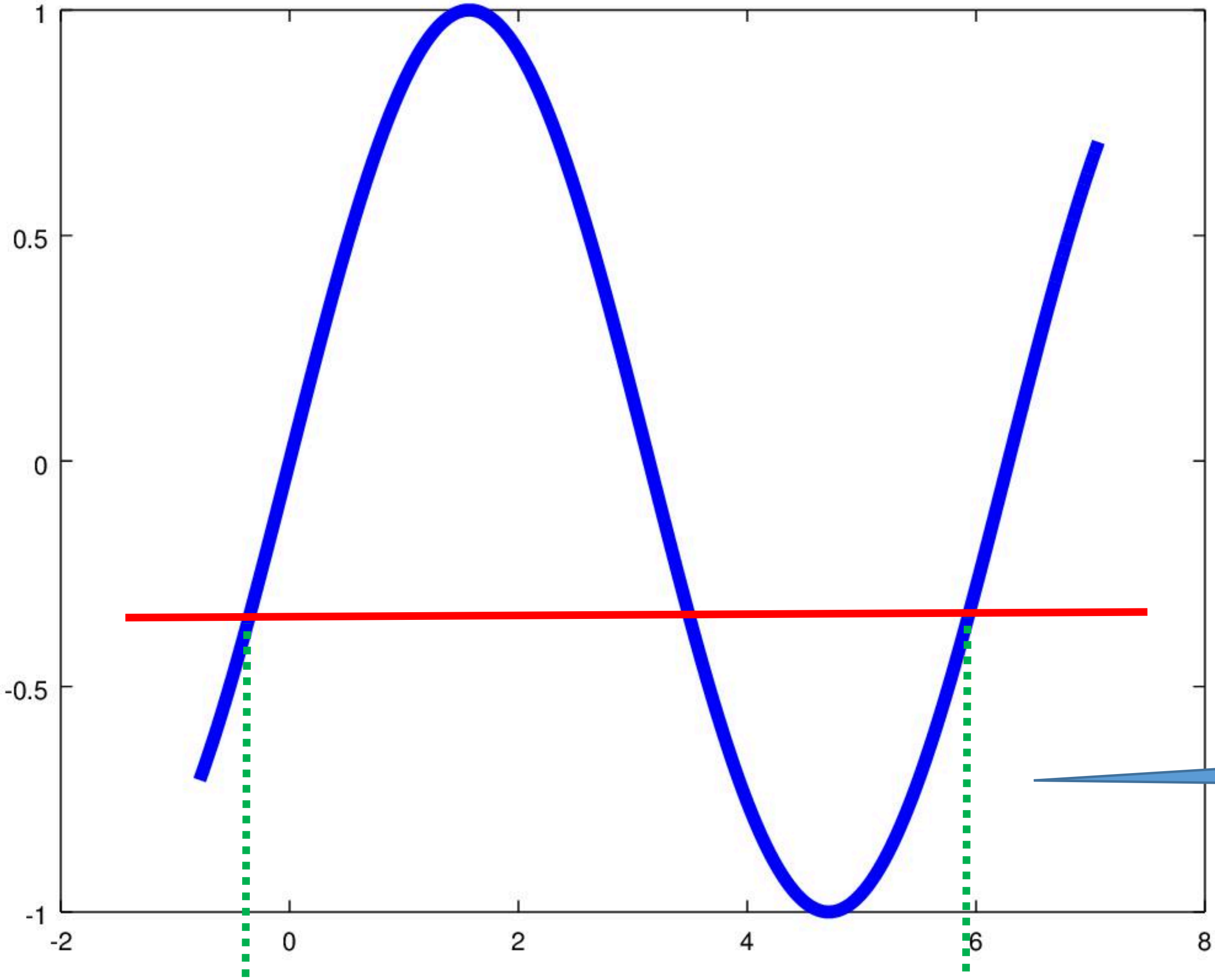
Convex Functions

```
x = -pi/4:pi/100:(9/4)*pi;
y = sin(x);
plot(x,y,'LineWidth',5)
```

Convex Functions



Convex Functions



Nonconvex function

$$\alpha f(x^1) + (1-\alpha) f(x^2) \geq f(\alpha x^1 + (1-\alpha) x^2)$$

Function NOT always below

Convex Functions

Gradient Test for Convex Functions: Suppose that X is a convex set and that the function, $f(x)$, is once continuously-differentiable on X. $f(x)$ is convex on X if and only if for any $\hat{x} \in X$ we have that:

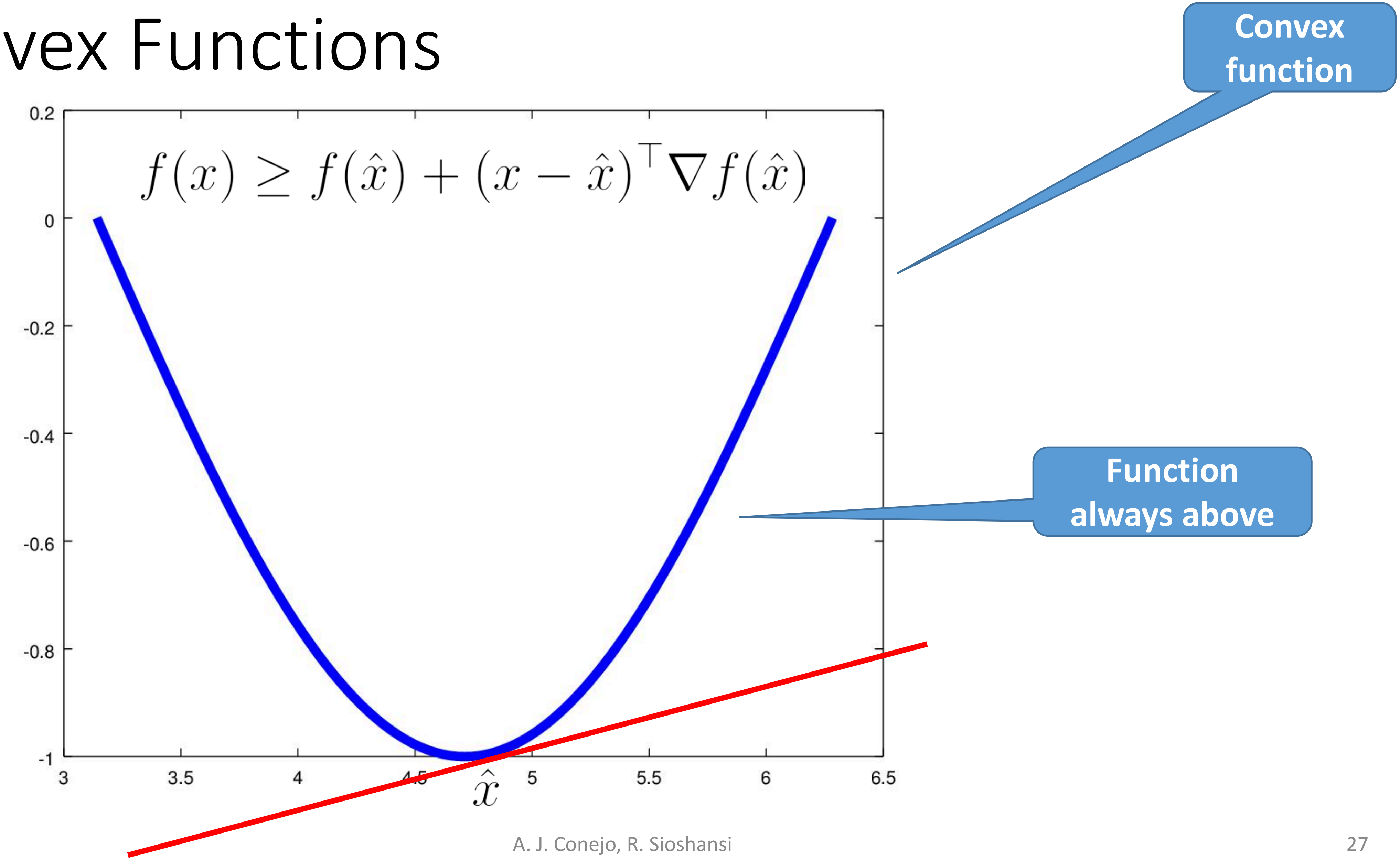
$$f(x) \geq f(\hat{x}) + (x - \hat{x})^{\top} \nabla f(\hat{x}),$$

for all $x \in X$.

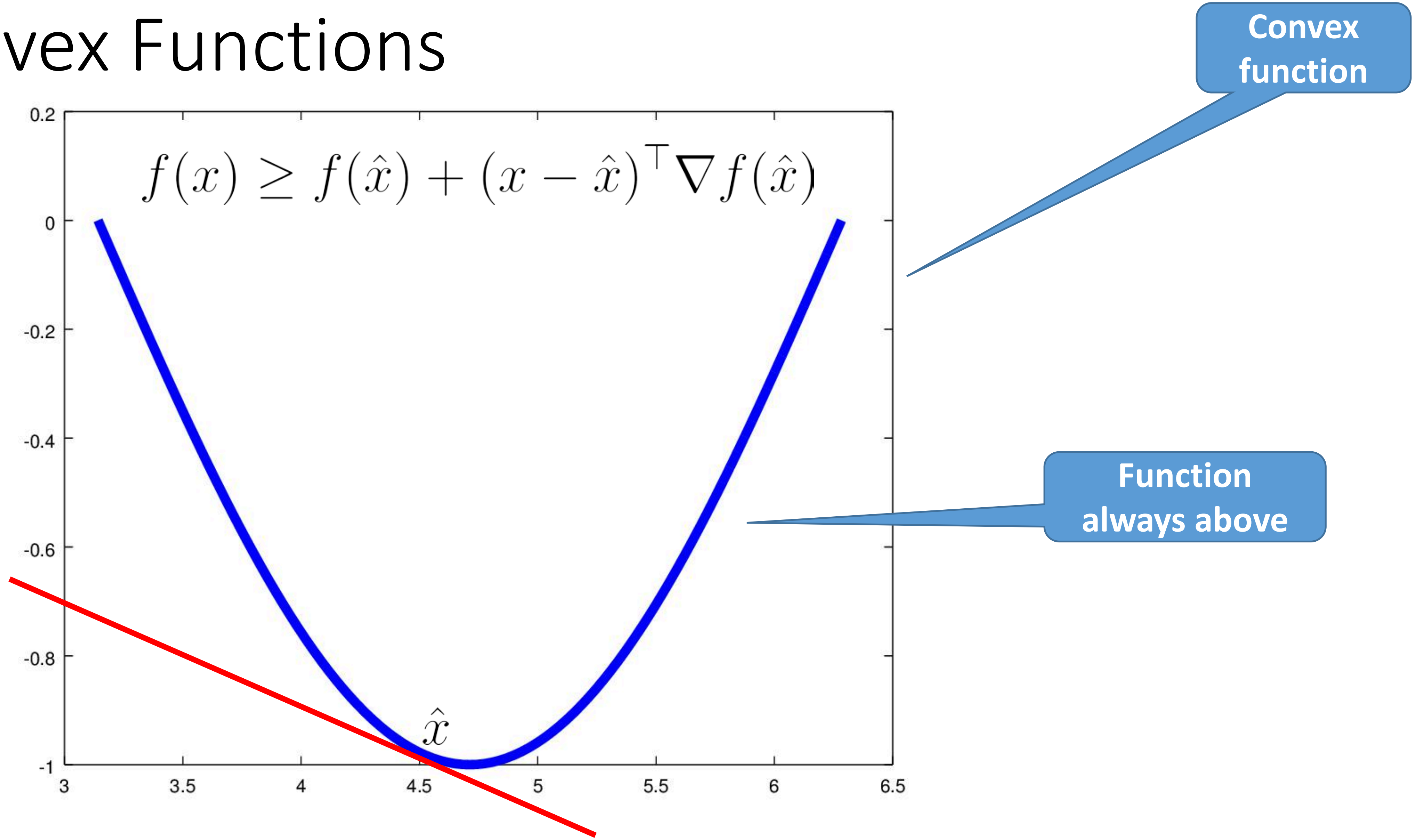
Convex Functions

$$f(x) \geq f(\hat{x}) + (x - \hat{x})^\top \nabla f(\hat{x})$$

Convex Functions

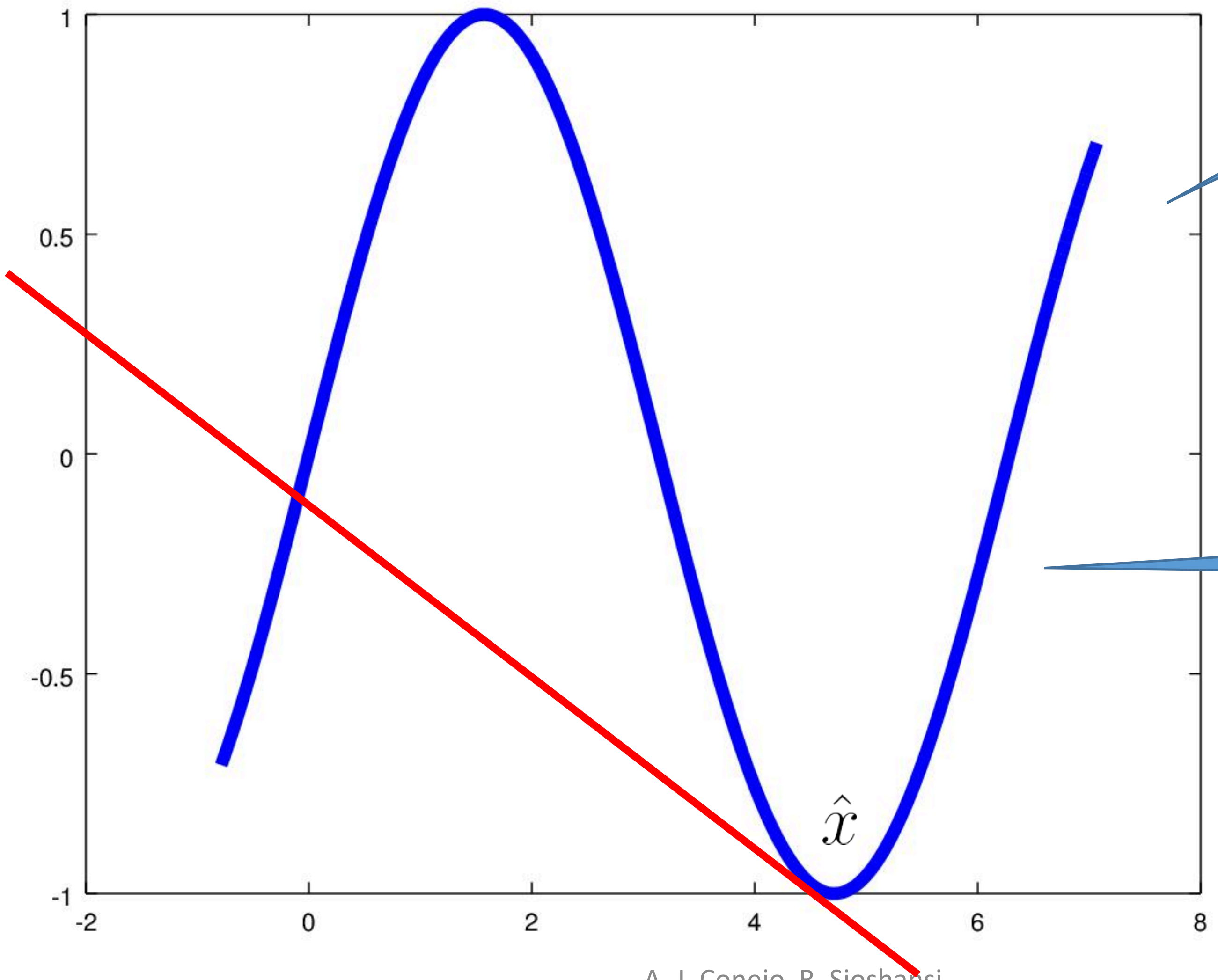


Convex Functions



$$f(x) \geq f(\hat{x}) + (x - \hat{x})^\top \nabla f(\hat{x})$$

Convex Functions



Nonconvex function

Function NOT always above

Convex Functions

Hessian Test for Convex Functions: Suppose that X is a convex set and that the function, $f(x)$, is twice continuously-differentiable on X. $f(x)$ is convex on X if and only if $\nabla^2 f(x)$ is positive semidefinite for any $x \in X$.

Convex Functions

Consider the function:

$$f(x) = x_1^2 + x_2^2,$$

We have that:

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

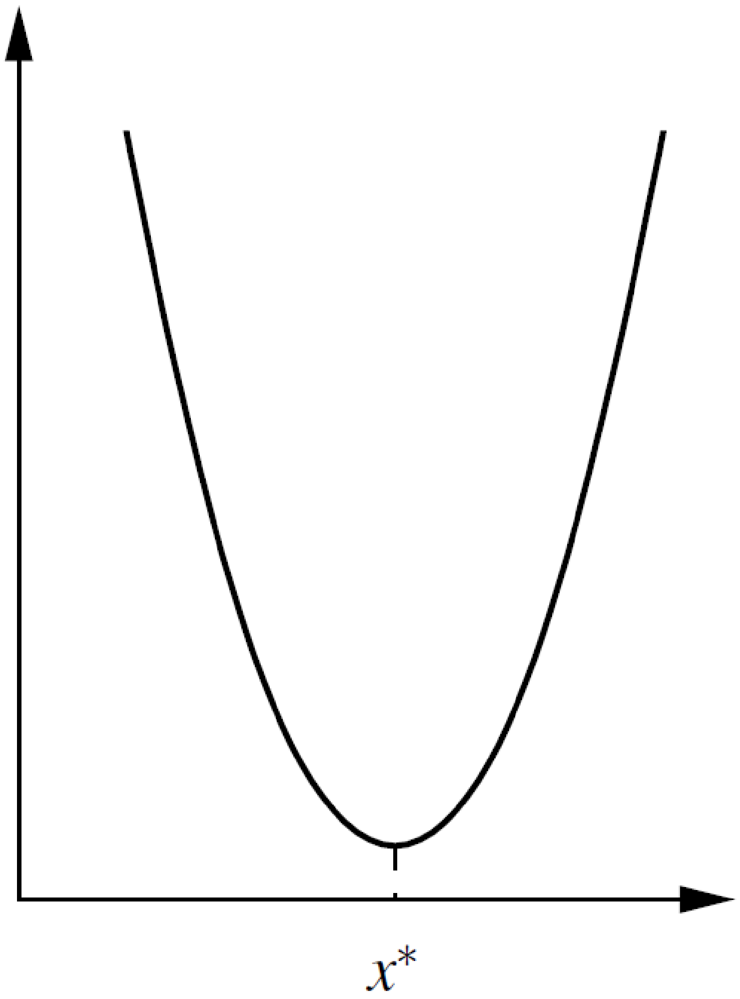
which is positive definite (and, thus, positive semidefinite) for any choice of x.

Global and Local Minima

Global Minima

Given a nonlinear optimization problem, a feasible solution, x^*, is a **global minimum** if $f(x^*) \leq f(x)$ for all other feasible values of x.

Global Minima

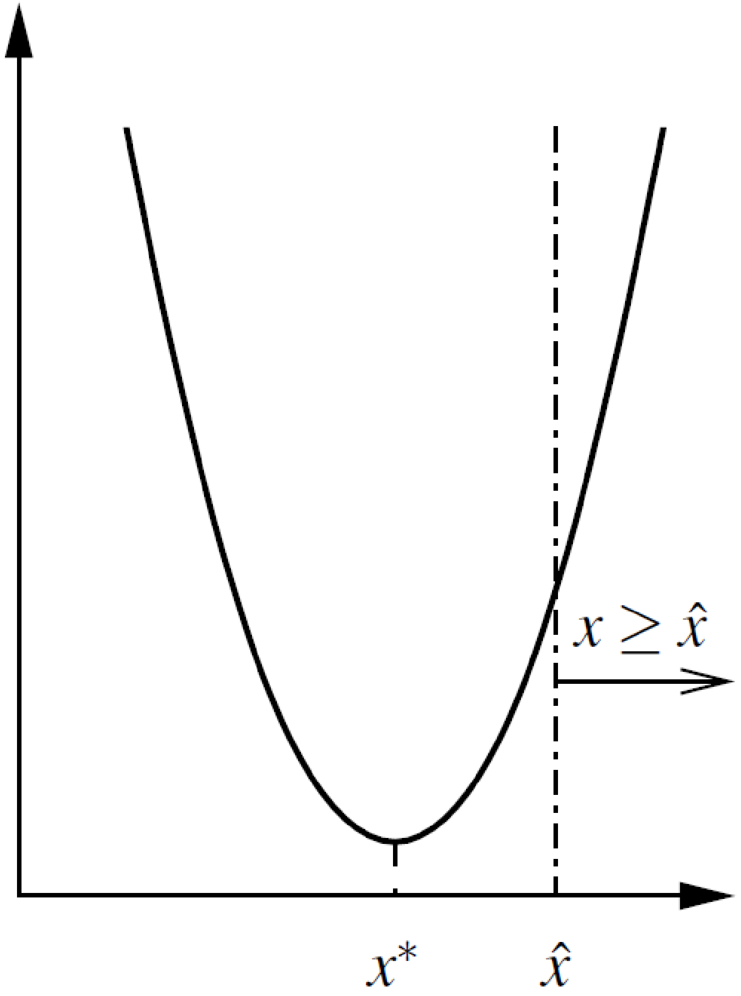


Global Minima

The figure below demonstrates the effect of adding the inequality constraint, $x \geq \hat{x}$, to the optimization problem illustrated in that figure.

Although x^* still gives the smallest objective-function value, it is no longer feasible. Thus, it does not satisfy the definition of a global minimum.

Global Minima



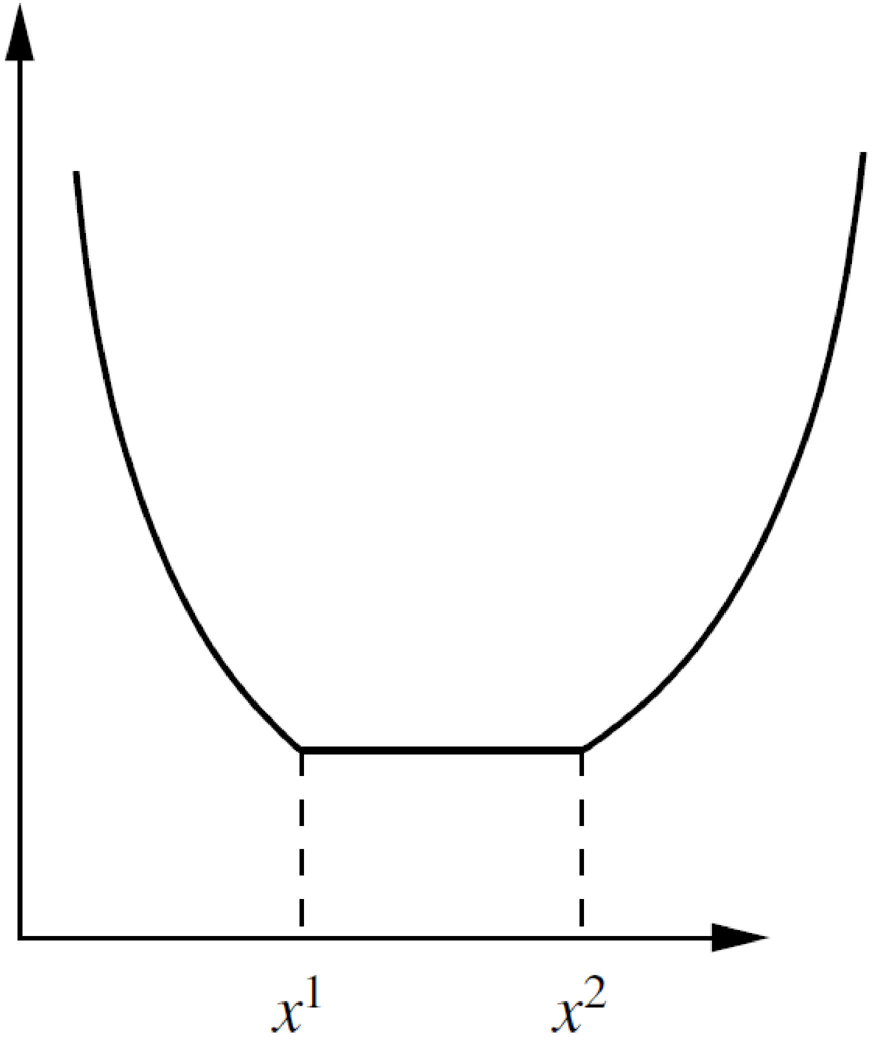
Global Minima

One can tell from visual inspection that $\hat{x}$ is in fact the global minimum of the parabolic objective function when the constraint is added.

Global Minima

It is also important to note that a nonlinear problem can have multiple global minima. This is illustrated in the figure below, where we see that all values of x between x^1 and x^2 are global minima.

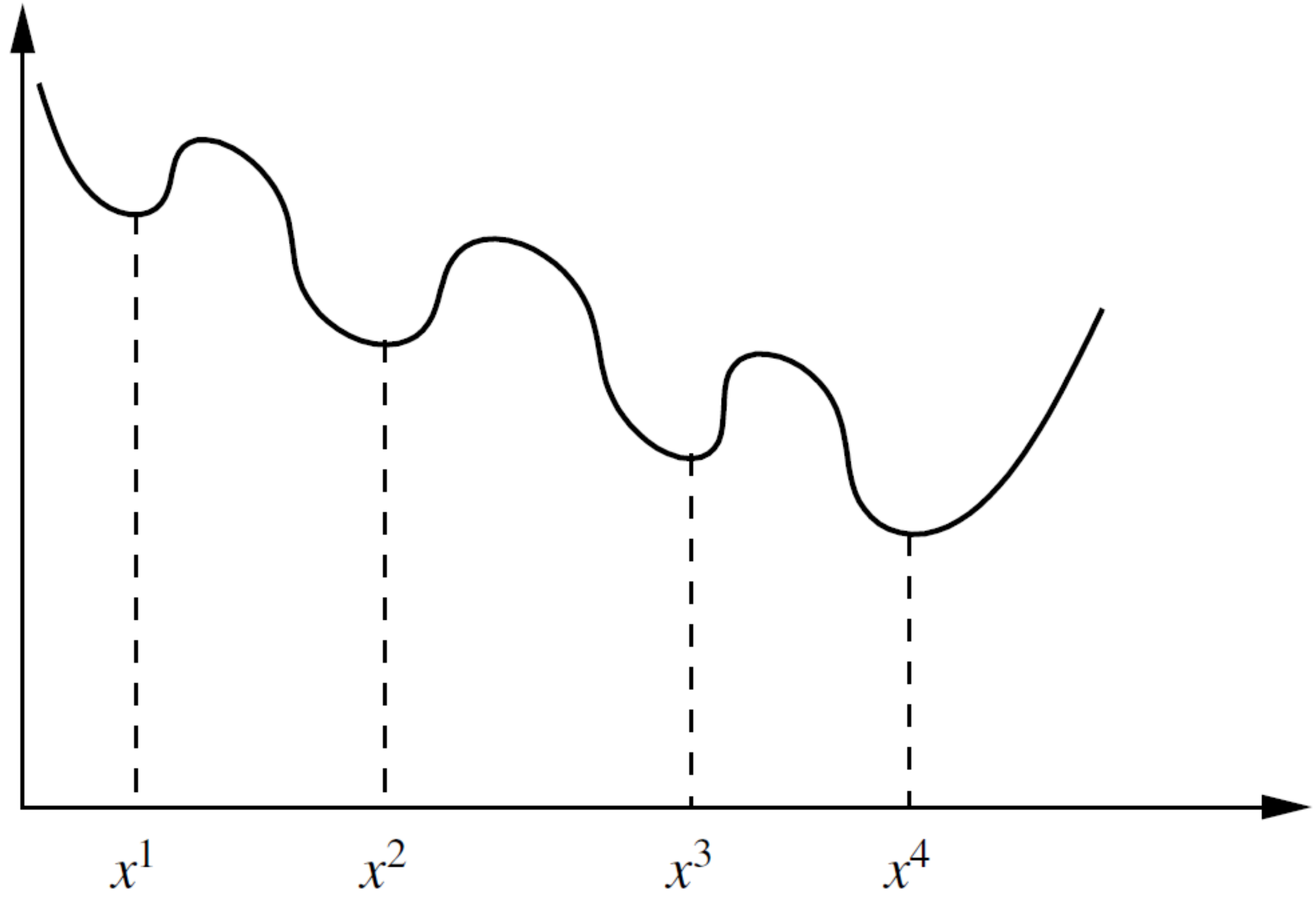
Global Minima



Local Minima

Given a nonlinear optimization problem, a feasible solution, x^*, is a **local minimum** if $f(x^*) \leq f(x)$ for all other feasible values of x that are close to x^*.

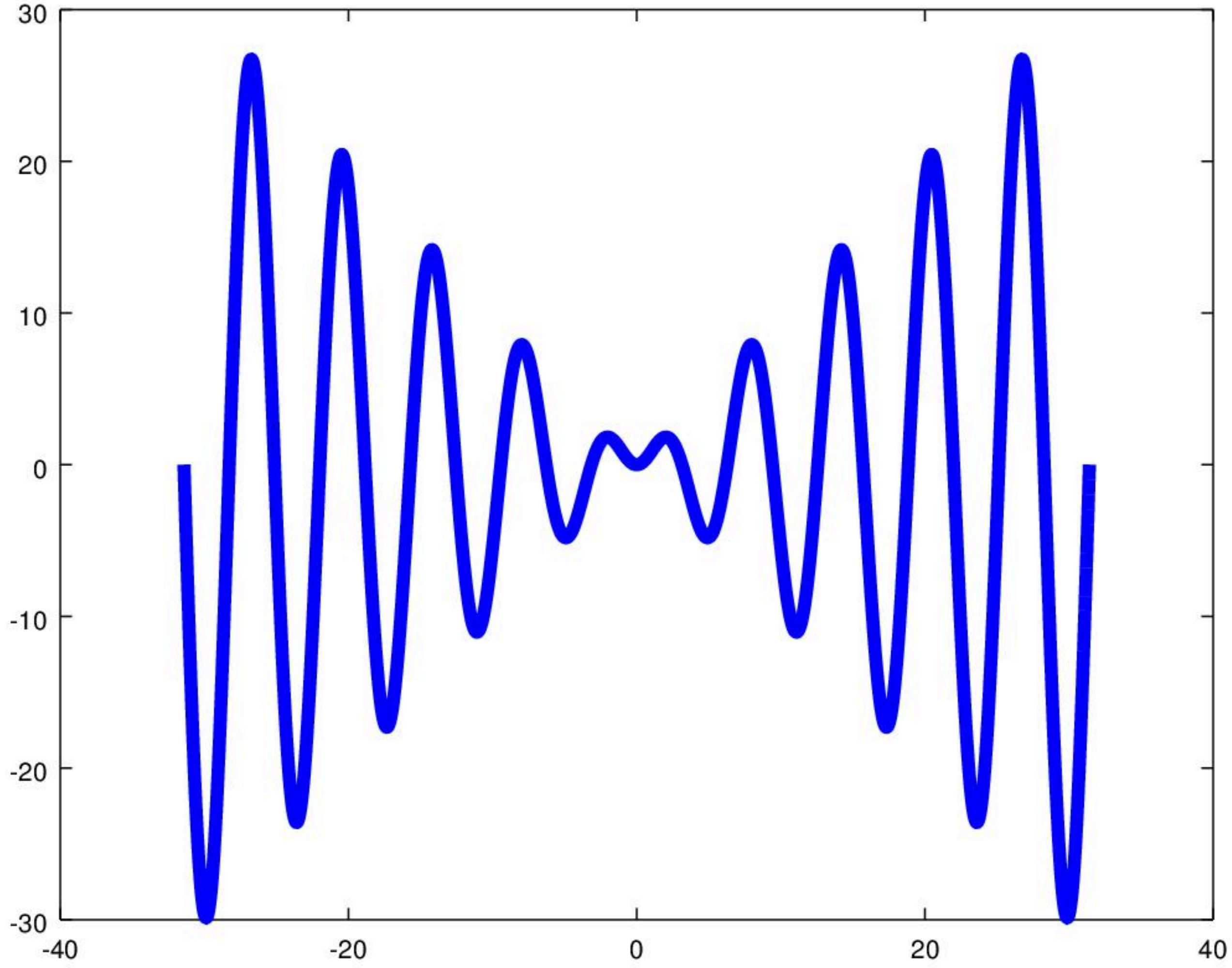
Local Minima



Local Minima

```
x = -10*pi:pi/100:10*pi;
y = x.*sin(x);
plot(x,y,'LineWidth',5)
```

Local Minima

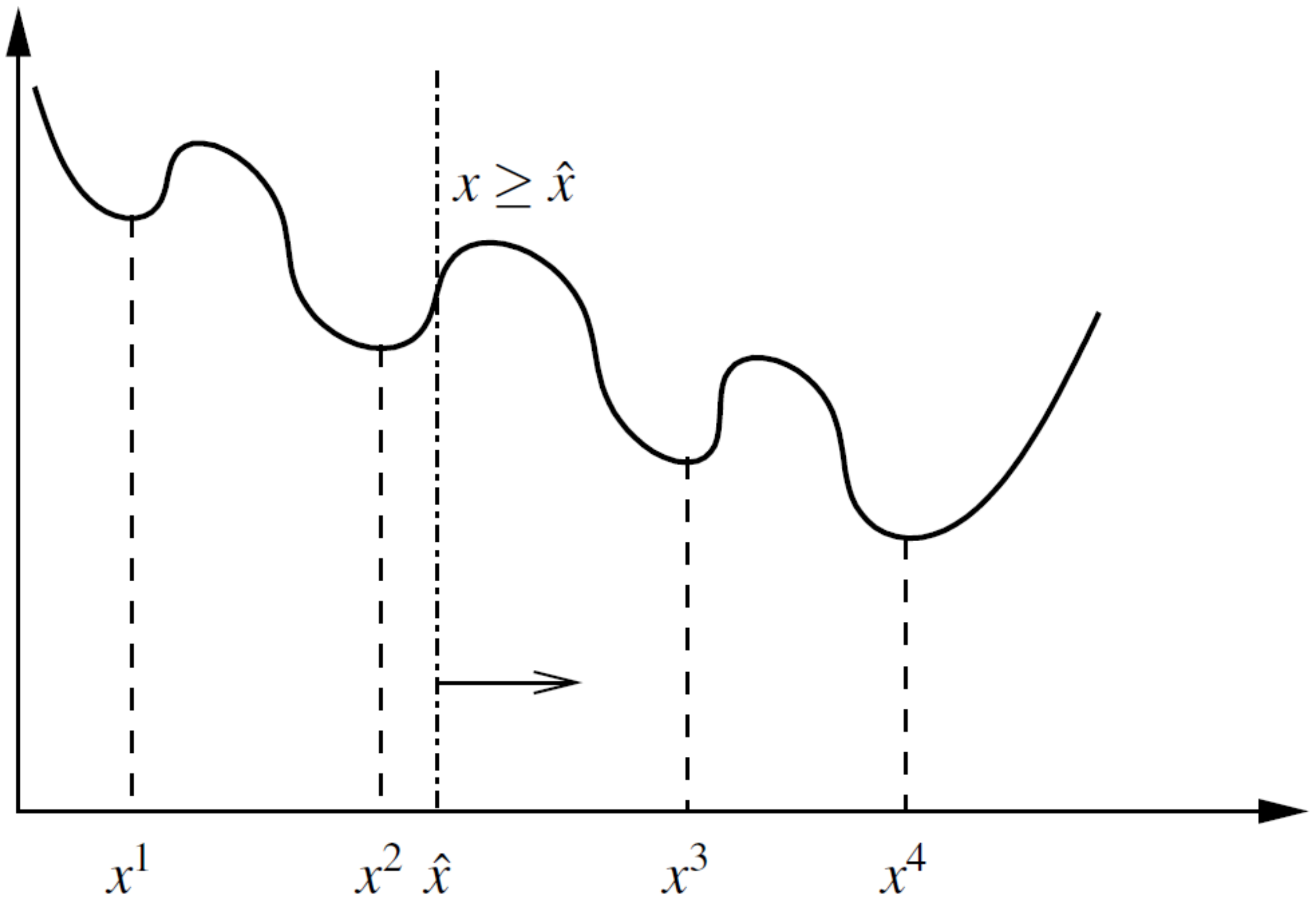


Local Minima

The figure below demonstrates how adding a constraint affects the definition of a local minimum. Here we have added the constraint $x \geq \hat{x}$.

Note that x^3 and x^4 are local minima and x^4 is the global minimum. However, x^1 and x^2 are not local minima, because they are no longer feasible.

Local Minima



Local Minima

Moreover, $\hat{x}$ is a local minimum.

To see why, note that x values that are close to $\hat{x}$ but to its left give smaller objective-function values than $\hat{x}$ does. However, these points are not considered in the definition of a local minimum, because we only consider *feasible* points that are close to $\hat{x}$.

If we restrict attention to feasible points that are close to $\hat{x}$ (i.e., points to the right of $\hat{x}$) then we see that $\hat{x}$ does indeed satisfy the definition of a local minimum.

Convex Optimization Problem

Convex Optimization Problem

An optimization problem of the form:

$$\min_{x} \; f(x)$$
$$\text{s.t.} \quad x \in X \subseteq \mathbb{R}^n,$$

is a **convex optimization problem** if the set X is convex and $f(x)$ is a convex function on the set X.

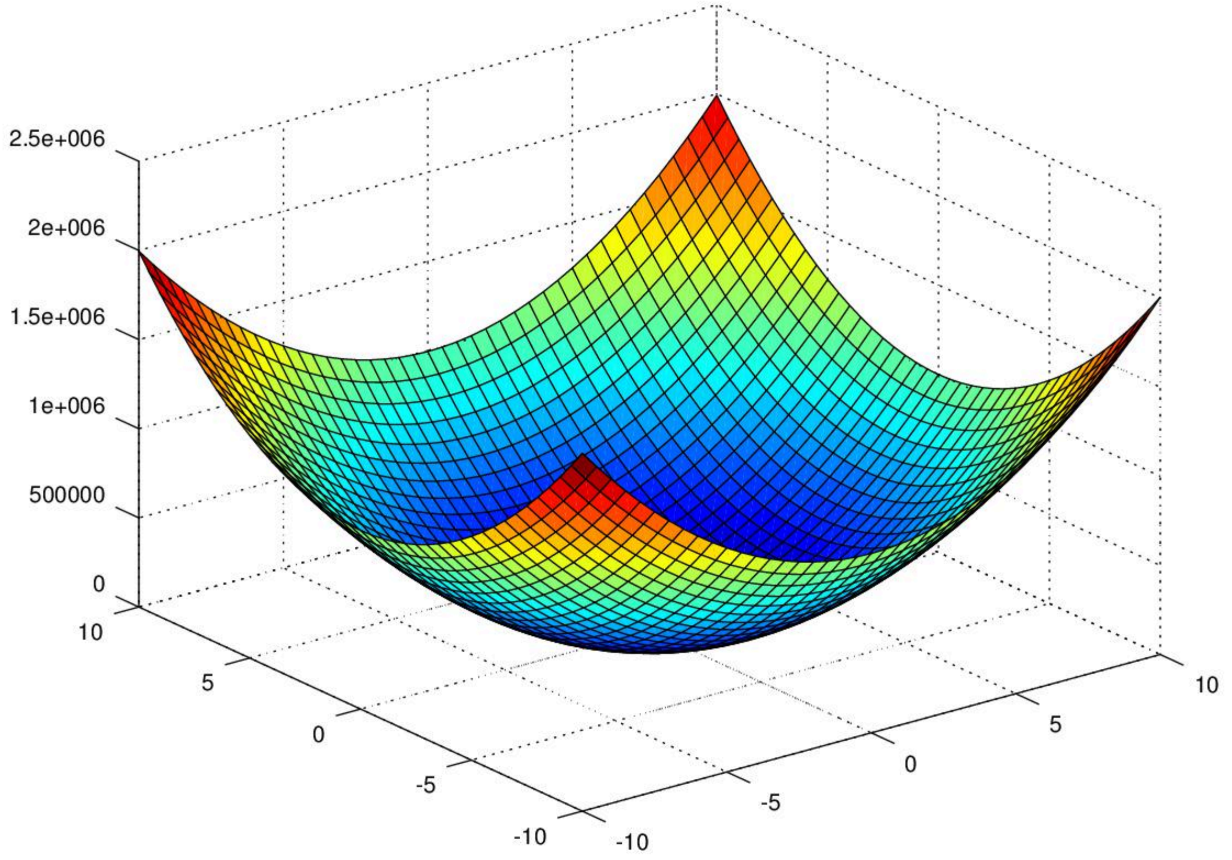
Convex Optimization Problem

Consider an optimization problem of the form:

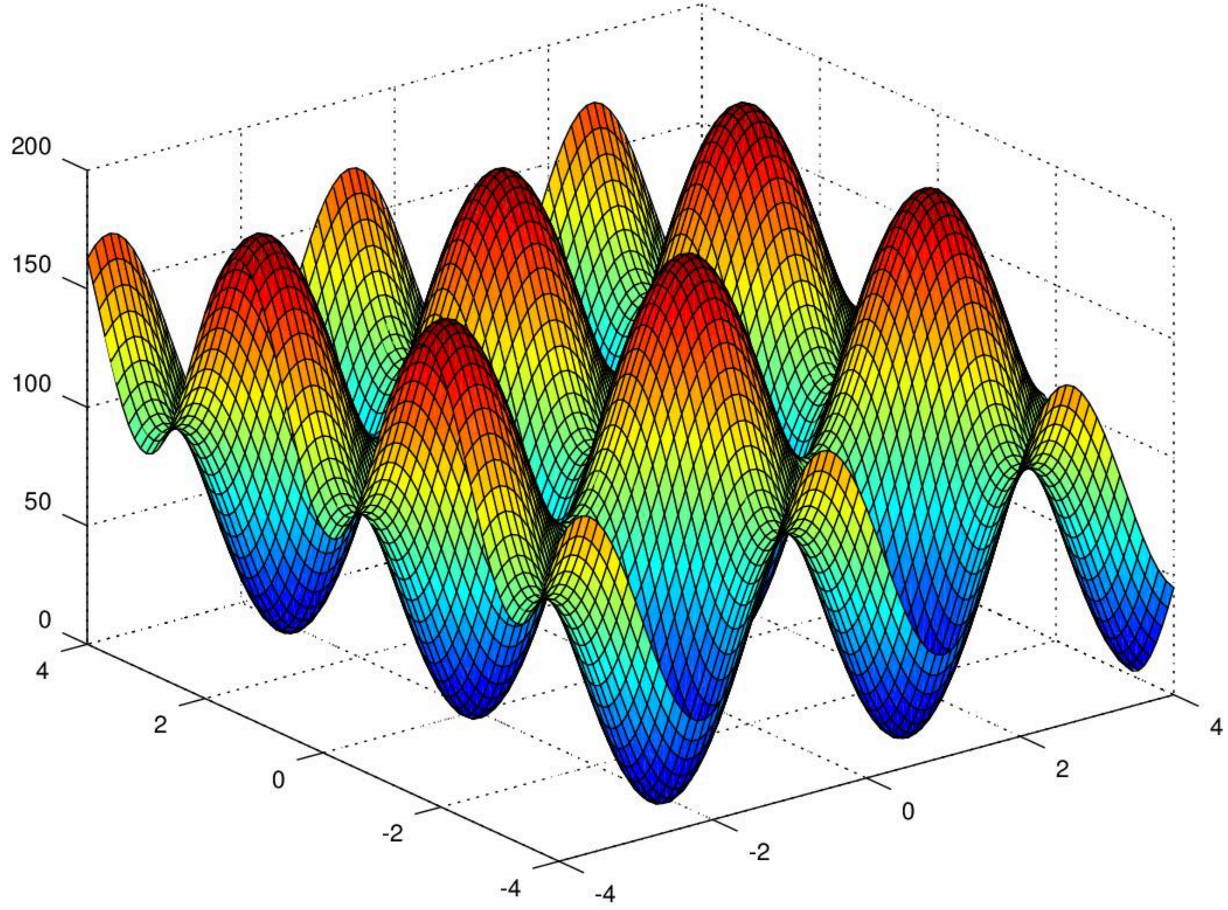
$$
\begin{aligned}
\min_{x} \ & f(x) \\
\text{s.t.} \ & x \in X \subseteq \mathbb{R}^n,
\end{aligned}
$$

where X is a convex set and $f(x)$ is a convex function on the set X. Any local minimum of this problem is also a global minimum.

Summary



Summary



This is it!