Math 2153 Dec. 1

Chapter 15 §15.4 – §15.6

• Framework of chapter 15

§15.5 <1> New Definitions: For a vector field $\vec{F} = \langle f, g, h \rangle$

① Divergence: $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

② Curl: $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})\vec{i} - (\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z})\vec{j} + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\vec{k}$

<2> New integrals

§15.2 ① Line integral: assuming C has parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a \le t \le b$

Scalar: $\int_C f\, ds = \int_a^b f(x(t), y(t), z(t))\, |\vec{r}'(t)|\, dt$

Vector Field: $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t)\, dt$

§15.6 ② Surface integral: assuming S has parameterization $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

or can be expressed as $z = z(x,y)$

Type I: $\iint_S f\, ds = \iint_R f(x(u,v), y(u,v), z(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\, dA$: R is domain of u, v

$\iint_S \vec{F} \cdot \vec{n}\, ds = \iint_R \vec{F} \cdot (\pm\, \vec{r}_u \times \vec{r}_v)\, dA$. "$\pm$" depends on direction of $\vec{n}$

Type II: $\iint_S f \cdot ds = \iint_R f(x, y, z(x,y)) \cdot \sqrt{1 + z_x^2 + z_y^2}\, dA$

(special case)

$\iint_S \vec{F} \cdot \vec{n}\, ds = \iint_R \vec{F} \cdot (\pm \langle -z_x, -z_y, 1 \rangle) \cdot dA$ "$\pm$" depends on direction of $\vec{n}$

<3> Three theorems

§15.4 Green's Theorem: 2D Line integral $\longleftrightarrow$ Double integral

Thm: C is a closed counterclockwise curve that encloses a simple connected domain;

$\vec{F} = \langle f, g \rangle$ is a vector field and f, g has continuous derivatives.

Then $\oint_C \vec{F} \cdot d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\, dA$

$\oint_C \vec{F} \cdot d\vec{n} = \oint_C f\,dy - g\,dx = \iint_R (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y})\, dA$

Applications:

① If line integral is difficult to calculate, then we can calculate a related double integral if all conditions in the thm are satisfied;

② If a double integral is difficult to calculate, then we may change it to a line integral.

Special: Area $= \iint_R 1\, dA = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

§15.7 Stokes' Theorem: 3D line integral $\leftrightarrow$ surface integral

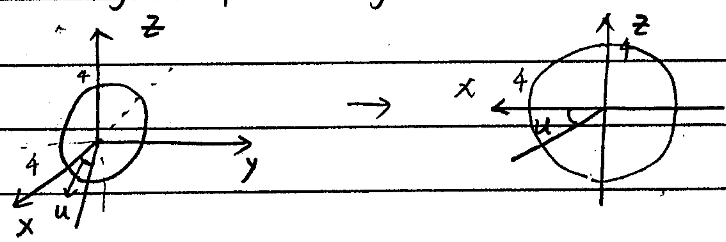
§15.8 Divergence Theorem: Surface integral $\leftrightarrow$ Triple integral.

HW12 Examples:

#9: $\iint_S f\, dS$ $\quad f = 4x$ and S is cylinder $x^2 + z^2 = 16$ for $0 \le y \le 1$.

Parameterization of S: $v = y$ and u is the angle in xz-plane from positive x-axis and increasing in a positive right hand rotation around y-axis

Answer:



$x = 4\cos u$

$z = 4\sin u$.

So $\vec{r}(u,v) = \langle 4\cos u,\ v,\ 4\sin u \rangle \qquad 0 \le u \le 2\pi \qquad 0 \le v \le 1$

$\vec{r}_u = \langle -4\sin u,\ 0,\ 4\cos u \rangle$

$\vec{r}_v = \langle 0,\ 1,\ 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -4\sin u & 0 & 4\cos u \\ 0 & 1 & 0 \end{vmatrix} = -4\cos u\, \vec{i} - 4\sin u\, \vec{k}$$

$|\vec{r}_u \times \vec{r}_v| = 4$

$$\iint_S f\, dS = \iint_R 4\cdot(4\cos u)\cdot 4\, dA$$

$$= \int_0^{2\pi}\int_0^1 64\cos u\, dv\, du$$

$$= 0$$

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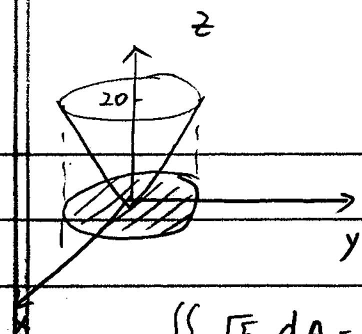
#10 $\iint_S 1\, dS$ where S is a cone $z^2 = 4(x^2+y^2)$ for $0 \le z \le 20$

Answer: $z = 2\sqrt{x^2+y^2}$ since $z \ge 0$

$z_x = \dfrac{2x}{\sqrt{x^2+y^2}}, \quad z_y = \dfrac{2y}{\sqrt{x^2+y^2}}$

$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \dfrac{4x^2}{x^2+y^2} + \dfrac{4y^2}{x^2+y^2}} = \sqrt{5}$

2 $\quad \iint_S 1\, dS = \iint_R \sqrt{5}\, dA \quad$ R is the region for x, y. (Projection of S onto xy-plane)



$z = 20 \Rightarrow \quad 20^2 = 4(x^2+y^2)$

$100 = x^2+y^2$ Circle.

$\iint_R \sqrt{5}\, dA = \sqrt{5} \cdot \text{Area of projection} = \sqrt{5} \cdot \pi \cdot 10^2 = 100\sqrt{5}\pi$ #

#13. Find the flux of the vector field $\vec{F} = \langle -y, x, 1 \rangle$ across the cylinder $y = 4x^2$. for $0 \le x \le 1$, $0 \le z \le 4$. Normal vectors point in the general direction of positive y.

Parameterize the surface using $u = x$ and $v = z$

Answer: $x = u \quad y = 4x^2 = 4u^2 \quad z = v$

$\vec{r}(u,v) = \langle u, 4u^2, v \rangle \qquad 0 \le u \le 1 \quad 0 \le v \le 4$

$\vec{r}_u = \langle 1, 8u, 0 \rangle \qquad \vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 8u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8u\vec{i} - \vec{j} = \langle 8u, \underset{\substack{\uparrow \\ \text{Negative}}}{-1}, 0 \rangle$

Since the normal vector is in positive y direction

$\vec{F} \cdot (-\vec{r}_u \times \vec{r}_v) = \langle -4u^2, u, 1 \rangle \cdot \langle -8u, 1, 0 \rangle$

$= 32u^3 + u$

$\iint_S \vec{F} \cdot \vec{n}\, ds = \iint_R 32u^3 + u\, dA$

$= \int_0^1 \int_1^4 32u^3 + u\, dv\, du$

$= 34$ #

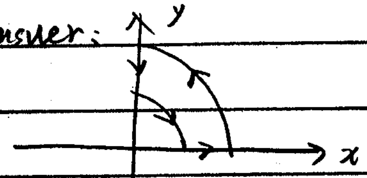
Applications of Green's theorem.

HW 11 #12 $\vec{F} = \langle 2x+y, x-4y \rangle$. R: quarter-annulus $\{(r,\theta) \mid 1 \le r \le 4, 0 \le \theta \le \frac{\pi}{2}\}$

C is the boundary of R.

Q: Find the circulation and flux of $\vec{F}$ along C.

Answer: $f = 2x+y \quad g = x - 4y$



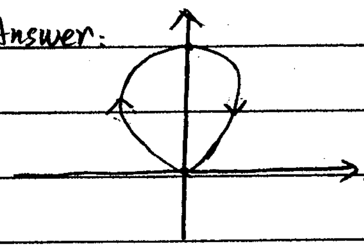
Circulation $= \oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (1-1)\, dA = 0$

Flux $= \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R (2-4)\, dA = -2$ Area of annulus

$= -\frac{15}{2}\pi$

Textbook §15.4 #22. Find the area of region bdd by $\vec{r}(t) = < t(1-t^2), 1-t^2 >$ $-1 \le t \le 1$.

Answer:



t ranges from -1 to 1, the curve is plotted clockwise!
To use Green's Theorem, need add a negative sign in the line integral!

$$\text{Area} = \iint_R 1\, dA = -\oint_C x\, dy = -\int_{-1}^{1} t(1-t^2)\cdot(1-t^2)'\, dt$$

$$= -\int_{-1}^{1} (t-t^3)\cdot(-2t)\, dt$$

$$= \frac{8}{15}$$

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