Chapter 15 Review

- We will focus on the three theorems in chapter 15. Before we solve any problems related to these thms, we will briefly review the basic tools which are needed in these thms.

Basic tools: line integral, surface integral and triple integral

Topic 1: Line integral (§15.2)

Suppose the parametrization of the curve is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a \le t \le b$

Case 1: Scalar function $\int_C f\,ds = \int_a^b f(x(t), y(t), z(t)) \cdot |r'(t)|\,dt$

$$= \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Case 2: Vector field $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}'(t)\,dt$

$$= \int_a^b f \cdot x' + g \cdot y' + h \cdot z'\,dt \qquad \vec{F} = \langle f, g, h \rangle$$

Summary: Parametrization of general curves

① Line segment from (a_1, b_1, c_1) to (a_2, b_2, c_2): $\vec{r}(t) = (a_1, b_1, c_1) + \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle t$ and $0 \le t \le 1$

Special cases: ex: $(0,0,0) \to (0,0,a)$ $\vec{r}(t) = \langle 0, 0, t \rangle$ $0 \le t \le a$

$(0,0,0) \to (a,a,a)$ $\vec{r}(t) = \langle t, t, t \rangle$ $0 \le t \le a$

② Circle $x^2 + y^2 = a^2$ counterclockwise

in 2D $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$ $0 \le t \le 2\pi$

in 3D $\vec{r}(t) = \langle a\cos t, a\sin t, 0 \rangle$ $0 \le t \le 2\pi$

If the direction is clockwise $\vec{r}(t) = \langle a\cos t, -a\sin t \rangle$ $0 \le t \le 2\pi$

③ Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ counterclockwise in 2D

$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$ $0 \le t \le 2\pi$

④ General curve $y = f(x)$ in 2D $\vec{r}(t) = \langle t, f(t) \rangle$

Example: §15.2 #46

Find the work to move an object using $\vec{F}$ on line segment from $(1,1,1)$ to $(8,4,2)$

$\vec{F} = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$

Answer: $\vec{r}(t) = (1,1,1) + (8-1, 4-1, 2-1)t = (1,1,1) + (7,3,1)t = \langle 7t+1, 3t+1, t+1 \rangle$

$$\vec{F}(t) \cdot \vec{r}'(t) = \frac{\langle 7t+1, 3t+1, t+1 \rangle}{(7t+1)^2 + (3t+1)^2 + (t+1)^2} \cdot \langle 7, 3, 1 \rangle = \frac{59t + 11}{(7t+1)^2 + (3t+1)^2 + (t+1)^2}$$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \frac{59t+11}{(7t+1)^2+(3t+1)^2+(t+1)^2}\,dt$ ⟸ u-substitution!

$u = (7t+1)^2+(3t+1)^2+(t+1)^2$ $\quad du = [2(7t+1)\cdot 7 + 2(3t+1)\cdot 3 + 2(t+1)]dt = (118t+22)dt$

$= \int_3^{84} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln u\Big|_3^{84} = \frac{1}{2}\ln\frac{84}{3} = \frac{1}{2}\ln 28 = \ln(2\sqrt{7})$ #

Topic 2 Surface Integral. (Table 15.3 in §15.6)

Case 1. f is a scalar function

Subcase 1. Parametrize the surface S, $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Then $\iint_S f\,ds = \iint_R f(u,v)\cdot|\vec{r}_u\times\vec{r}_v|\,dA$ R is the area for u and v

Subcase 2. S has explicit equation $z = z(x,y)$

Then $\iint_S f\,ds = \iint_R f(x,y,z(x,y))\cdot\sqrt{1+z_x^2+z_y^2}\,dA$. R is the area for x and y (projection of S onto xoy plane)

Case 2. $\vec{F} = \langle f, g, h\rangle$ is a vector field

Subcase 1. Parametrize the surface S. $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Then $\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \vec{F}\cdot(\pm\,\vec{r}_u\times\vec{r}_v)\,dA$ the choice of sign "±" depends on the direction of normal vector.

Subcase 2: S has explicit equation $z = z(x,y)$

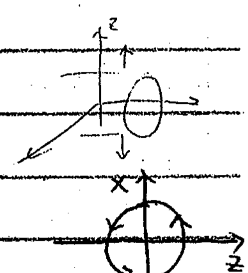
Then $\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \vec{F}\cdot(\pm\langle -z_x, -z_y, 1\rangle)\cdot dA$. R is the projection of S.

Note: In case 1 (scalar case), if $f\equiv 1$, $\iint_S 1\,ds$ = Area of surface S

Example §15.6 #53

$\iint_S \frac{\langle x, 0, z\rangle}{\sqrt{x^2+z^2}}\cdot n\,ds$ $\quad S: x^2+z^2=a^2 \quad -2\le y\le 2$ normal vector: outward

Answer: Case 2 subcase 1



$\vec{r}(u,v) = \langle a\sin u, v, a\cos u\rangle \quad 0\le u\le 2\pi \quad -2\le v\le 2$

$\vec{r}_u = \langle a\cos u, 0, -a\sin u\rangle \quad \vec{r}_v = \langle 0, 1, 0\rangle$

$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} i & j & k \\ a\cos u & 0 & -a\sin u \\ 0 & 1 & 0\end{vmatrix} = a\sin u\,\vec{i} + a\cos u\,\vec{k}$

$= \langle a\sin u, 0, a\cos u\rangle$

$\vec{F}(u,v) = \langle \frac{a\sin u}{a}, 0, \frac{a\cos u}{a}\rangle = \langle \sin u, 0, \cos u\rangle$

$\vec{F}\cdot(\vec{r}_u\times\vec{r}_v) = \langle a\sin u, 0, a\cos u\rangle\cdot\langle\sin u, 0, \cos u\rangle = a$

review exercise

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi}\int_{-2}^{2} a\,dv\,du = 8\pi a \qquad \#$$

P.184 #46 Find surface area of cone $z^2=x^2+y^2$, $2\le z\le 4$ (No bases)

Answer: $S=\iint_S 1\,ds$ $\quad$ S: $z=\sqrt{x^2+y^2}$ since $z\ge 0$ $\quad$ Case 1 subcase 2

$=\iint_R \sqrt{1+z_x^2+z_y^2}\,dA$ $\quad$ R: $4\le x^2+y^2\le 16$ (Projection

$z_x=\frac{x}{\sqrt{x^2+y^2}}$ $\quad z_y=\frac{y}{\sqrt{x^2+y^2}}$ $\quad 1+z_x^2+z_y^2 = 1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2} = 2$

$=\iint_R \sqrt{2}\,dA = \sqrt{2}\cdot$ Area of torus $=\sqrt{2}\cdot(\pi\cdot 16-\pi\cdot 4) = 12\pi\sqrt{2}$

Note: Also try to parametrize S and use the formula in Case 1 sub case 1 #

Topic 3. Triple integral

Cartesian, Cylindrical, Spherical. Review by yourself

Topic 4. Green's thm.

Thm: C is a closed counterclockwise curve (1), that encloses a simply connected domain (2)

$\vec{F}=\langle f,g\rangle$, f and g have continuous derivatives (3)

Then $\oint_C \vec{F}\cdot d\vec{r} = \oint_C f\,dx+g\,dy = \iint_R \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dA$

$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy-g\,dx = \iint_R\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)dA$

Note: the conditions are important. See counterexamples in Lecture Notes

Application: By choosing special $\vec{F}$, we can use Green's thm to calculate area of region R

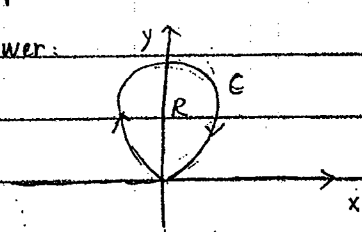
Area of R $=\iint_R 1\,dA$. So choose $F=\langle f,g\rangle$ such that $\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}=1$

ex: $\vec{F}=\langle 0, x\rangle$ $\quad \vec{F}=\langle -y,0\rangle$ or $\vec{F}=\langle -\frac{1}{2}y, \frac{1}{2}x\rangle$

So Area of R $=\iint_R 1\,dA = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy-y\,dx)$

Example: §15.4 #22. Find the area of region bdd by $\vec{r}(t)=\langle t(1-t^2), 1-t^2\rangle$ for $-1\le t\le 1$

Answer: $\quad$ C: closed, clockwise. $\quad$ R: simply connected



3 $\quad$ Area of $R=\iint_R 1\,dA = -\oint_C x\,dy = -\int_{-1}^{1} t(1-t^2)\cdot(1-t^2)'\,dt$

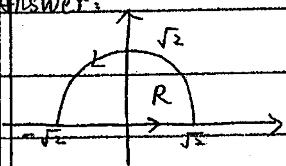
$= -\int_{-1}^{1}(t-t^3)\cdot(-2t)\,dt = 2\cdot\left(\frac{t^3}{3}-\frac{t^5}{5}\right)\Big|_{-1}^{1} = \frac{8}{15}$ #

review exercise

Example P1183 #30 counterclockwise

$\oint_C \underbrace{(-3y+x^{3/2})}_{f}\,dx + \underbrace{(x-y^{2/3})}_{g}\,dy$ C is the boundary of $\{(x,y)\ x^2+y^2 \le 2,\ y \ge 0\}$

Answer:



$\vec{F} = \langle f, g\rangle = \langle -3y+x^{3/2},\ x-y^{2/3}\rangle$ f, g have cts derivatives

R is simply connected

By Green's thm

$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (1+3)\,dA$

$= 4\ \text{Area of half disk} = 4 \cdot \frac{1}{2} \cdot \pi \cdot 2 = 4\pi$ #

Topic 5 Stokes' thm

Thm Let S be an oriented surface with boundary C, $\vec{F} = \langle f, g, h\rangle$ and f, g, h have cts derivatives on S. Then $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times F)\cdot\vec{n}\,dS$; $\vec{n}$ is the unit normal vector of S.

Note: The underlined condition is important. See counterexample #43 on page 1170

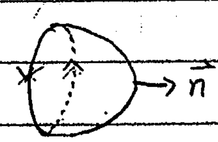
Example P1184 #60 review exercise

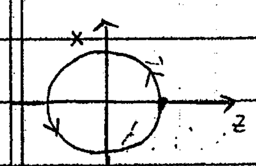
Use Stokes' thm to calculate $\iint_S (\nabla\times F)\cdot\vec{n}\,dS$

$\vec{F} = \langle x^2-z^2,\ y^2,\ xz\rangle$ S: hemisphere $x^2+y^2+z^2=4$ $y \ge 0$ $\vec{n}$ outward

Answer: By Stokes' thm $\iint_S (\nabla\times F)\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

C: $x^2+z^2=4$ and counterclockwise





$\vec{r}(t) = \langle 2\sin t,\ 0,\ 2\cos t\rangle$ $0 \le t \le 2\pi$

$\vec{r}'(t) = \langle 2\cos t,\ 0,\ -2\sin t\rangle$

$\vec{F} = \langle 4\cos^2 t - 4\sin^2 t,\ 0,\ 4\cos t\sin t\rangle$

$\vec{F}\cdot\vec{r}' = 8\cos^3 t - 8\sin^2 t\cos t + 8\cos t\sin^2 t$

$= 8\cos t - 8\sin^2 t\cos t$

$= 8\cos t(1-\sin^2 t) = 8\cos^3 t$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 8\cos^3 t\,dt = 0$ u-substitution ($u = \sin t$) #

Topic 6. Divergence Thm

Thm: $\vec{F} = \langle f, g, h\rangle$, f, g, h have cts derivatives. D is a simply connected region enclosed by an oriented surface S. Then $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV$ $\vec{n}$: unit normal.

Thm: Same condition on $\vec{F}$. D is a hallow region bdd by S_1 and S_2. Then

$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_D \nabla\cdot\vec{F}\,dV$

(Textbook §15.8 #26)

$\vec{F} = 2r \cdot |r| = 2\langle x, y, z\rangle \cdot \sqrt{x^2+y^2+z^2}$: D is the region between the spheres $r=2$ and $r=\sqrt{5}$

Calculate $\iint_S \vec{F}\cdot\vec{n}\,ds$ $S = S_1 \cup S_2$ Hallow region

Answer: $\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_D \nabla\cdot\vec{F}\,dv$.

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}f + \frac{\partial}{\partial y}g + \frac{\partial}{\partial z}h$. where $f = 2x\sqrt{x^2+y^2+z^2}$ $g = 2y\sqrt{x^2+y^2+z^2}$ $h = 2z\sqrt{x^2+y^2+z^2}$

$\nabla\cdot\vec{F} = 8\sqrt{x^2+y^2+z^2}$

$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 8\sqrt{x^2+y^2+z^2}\,dV$

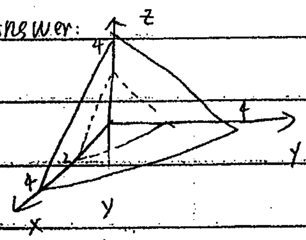
$= \int_0^{2\pi}\int_0^{\pi}\int_2^{\sqrt{5}} 8\rho\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$= 72\pi$ #

(Textbook §15.8 26)

$\vec{F} = \langle x^2, -y^2, z^2\rangle$ D is the region bdd by $z = 4-x-y$ $z = 2-x-y$ in 1st octant.

Answer:



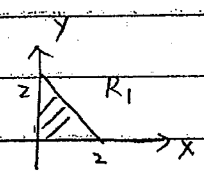
$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_D \nabla\cdot\vec{F}\,dV$

$\nabla\cdot\vec{F} = 2x - 2y + 2z = 2(x-y+z)$

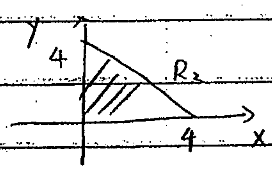
$\iiint_D \nabla\cdot\vec{F}\,dV = 2\iiint_D x-y+z\,dV$

$D = D_1 - D_2$ D_1: bigger tetrahedron

D_2 Smaller tetrahedron



$= 2\iiint_{D_1}(x-y+z)\,dV - 2\iiint_{D_2}(x-y+z)\,dV$



$= 2\iint_{R_2}\int_0^{4-x-y}(x-y+z)\,dz\,dA - 2\iint_{R_1}\int_0^{2-x-y}(x-y+z)\,dz\,dA$

$= 2\int_0^4\int_0^{4-x}\int_0^{4-x-y}(x-y+z)\,dz\,dy\,dx - 2\int_0^2\int_0^{2-x}\int_0^{2-x-y}(x-y+z)\,dz\,dy\,dx$

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Note:

① φ is a scaler function $\nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$

② $\vec{F}$ is a vector field $\underset{\text{curl}}{\nabla\times\vec{F}} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$ $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}f + \frac{\partial}{\partial y}g + \frac{\partial}{\partial z}h$

divergence