§15.7 — 15.8

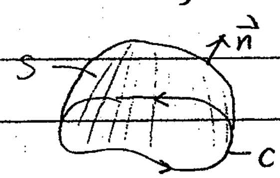
Topic 1: Stokes's Thm (§15.7)

Thm: Let S be an oriented surface in $\mathbb{R}^3$ with a piece-wise smooth closed boundary C whose orientation is consistent with that of S. Assume $\vec{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS \qquad \vec{n}: \text{unit normal vector of } S$$

Note:

① The orientations of C and S satisfy right hand rule.



② If the underlined condition is not satisfied, then the Stokes's thm does not hold! See problem 43 on page 1170.

· Applications:

① Using $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$ to calculate $\oint_C \vec{F} \cdot d\vec{r}$

Key point: C is given, choose a nice S with the boundary C and apply surface integral formula to $\nabla \times \vec{F}$ (Page 1156) Practice: 11-16 on Page 1168

② Using $\oint_C \vec{F} \cdot d\vec{r}$ to calculate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$

Key point: Since S is given, C is fixed. Apply line integral formula to $\vec{F}$ (Page 1100-1101) Practice: 17-20 on Page 1168

Example: P1168 Ex 13

$\vec{F} = \langle x^2 - z^2, y, 2xz \rangle$. C is the boundary of the plane $z = 4 - x - y$ in the first octant. C has a counterclockwise orientation.

Q: Evaluate $\oint_C \vec{F} \cdot d\vec{r}$

A: Method 1: Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ directly. Need to parametrize $\vec{C}$. Complex!

Method 2: Use Stokes' thm. Check conditions! All satisfied!

By thm, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$ Q: How do we choose S.

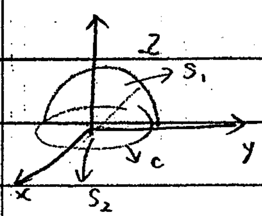
A: Choose S as simple as possible!

(b). Find A by choosing another surface and evaluate the surface integral.

Answer: $\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\, ds = \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\, ds$

S_1 and S_2 have the same boundary C!

But we can choose a simplier surface S_2



Choose S_2 as the disk $x^2+y^2 \le 1$ on xoy plane!

Now, we handle $\iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\, ds$. $\vec{n} = \langle 0, 0, 1\rangle$ easy to find!

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = \langle 2x, 0, -2z\rangle$$

So on S_2 ($z=0$!), $\nabla\times\vec{F} = \langle 2x, 0, 0\rangle$

$(\nabla\times\vec{F})\cdot\vec{n} = 0$

So $\iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\, ds = 0$ #

Topic 2: Divergence Thm (§15.8)

Thm: Let $\vec{F} = \langle f, g, h\rangle$ whose components have cts 1st derivatives in a simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\, ds}_{\text{Net flux}} = \underbrace{\iiint_D \nabla\cdot\vec{F}\, dv}_{\text{3D integral of } \nabla\cdot\vec{F}}$$

(Final exam) ~~counterexample~~ in lecture notes!

Note: If conditions ① and ② are not satisfied, the thm may not hold.

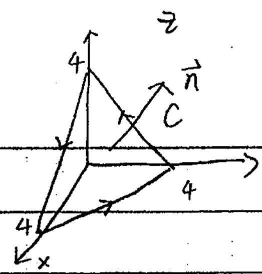
If D is a hollow region, then ② is not satisfied. Please check thm 15.18

Application: Using 3D integral to evaluate net flux. (Surface integral)

Ex. Page. 1180 #23

3 $\vec{F} = \langle x, y, z\rangle$ S: surface of paraboloid $z = 4-x^2-y^2$ $z \ge 0$. plus its base on xoy plane

Q: Find net flux $\iint_S \vec{F}\cdot\vec{n}\, ds$



Choose S as the plane in the 1st octant !

Next, we handle $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds$: surface integral for $\nabla\times\vec{F}$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-z^2 & y & 2xz \end{vmatrix} = 0\cdot\vec{i} - 4z\vec{j} + 0\cdot\vec{k}$$

$$= \langle 0, -4z, 0\rangle$$

How do we evaluate $\iint_S \langle 0,-4z,0\rangle\cdot\vec{n}\,ds$?

It's type III ! and $z = 4-x-y$ has explicit expression !

So $\iint_S \langle 0,-4z,0\rangle\cdot\vec{n}\,ds$

choose positive, since $\vec{n}$ is upward !

Formula: $\iint_R \langle 0, -4(4-x-y), 0\rangle\cdot(\pm\langle -z_x, -z_y, 1\rangle)\,dA$

$\iint_R \langle 0, -4(4-x-y), 0\rangle\cdot\langle 1,1,1\rangle\,dA$ $\quad$ R is the projection on xoy plane

$$= \int_0^4\int_0^{4-x} -4(4-x-y)\,dy\,dx$$

$$= -\frac{128}{3}$$

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Example 2. Page 1170 #41

Evaluate $A = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds$. $\vec{F} = \langle yz, -xz, xy\rangle$ S: upper half of ellipsoid

$x^2+y^2+8z^2=1 \quad z\geq 0$

(a) Evaluate it using line integrals.

Answer: $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds = \oint_C \vec{F}\cdot d\vec{r}$. C is $x^2+y^2=1$ counterclockwise

Now, we handle $\oint_C \vec{F}\cdot d\vec{r}$

$\vec{r}(t) = \langle \cos t, \sin t, 0\rangle \quad 0\leq t\leq 2\pi$ $\quad$ parametrization of C.

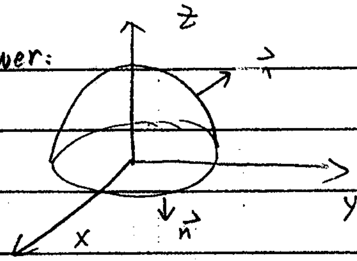
$\vec{F}(t) = \langle 0, 0, \cos t\sin t\rangle$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\vec{r}'(t)\,dt = \int_0^{2\pi} \langle 0,0,\cos t\sin t\rangle\cdot\langle -\sin t, \cos t, 0\rangle\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$

2

Answer:



$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_D (\nabla\cdot\vec{F})\, dv$$

$$\nabla\cdot\vec{F} = \tfrac{\partial}{\partial x}f + \tfrac{\partial}{\partial y}g + \tfrac{\partial}{\partial z}h = 1+1+1 = 3$$

$$\left.\begin{aligned}\iiint_D (\nabla\cdot\vec{F})\, dv &= \iiint_D 3\, dv = \iint_R \int_0^{4-x^2-y^2} 3\, dz\, dA \\ &= \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} 3\, dz\; r\,dr\,d\theta\end{aligned}\right\}\text{cylindrical coordinates!}$$

$$= \int_0^{2\pi}\int_0^{2} 3(4-r^2)\,r\, dr\,d\theta$$

$$= 24\pi \qquad \#$$

Suggested problems

Topic 1. Pages 1168–1170 : 11—20 41 43

Topic 2 Pages 1179–1180 : 17—30, 36 37