§15.6

§15.6 Surface integrals

- Recall:

1D integral : $\int_a^b f(x)\,dx$ or $\int_c^d f(y)\,dy$: Integral along straight line: (x-axis or y-axis)

2D integral: $\iint_R f(x,y)\,dA$. Integral on the region R in xy-plane.

3D integral $\iiint_V f(x,y,z)\,dV$: Integral on solid region V in xyz-space

Line integral : $\int_C f\,ds$ or $\int_C \vec{F}\cdot d\vec{r}$. C is a curve in space (2D or 3D)

Surface integral : Integral on a surface S in space. Three types!

Type I Surface integral of Scalar-Valued function on parameterized surfaces.

Suppose f is a scalar-valued function. $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $R = \{(u,v): a \le u \le b,\ c \le v \le d\}$ is a surface. The tangent vectors are

$\vec{t}_u = \frac{\partial r}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ and $\vec{t}_v = \frac{\partial r}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ Assume the normal vector $\vec{r}_u \times \vec{r}_v \ne \vec{0}$, then

$$\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))\,|\vec{t}_u \times \vec{t}_v|\,dA$$

Note: If $f \equiv 1$, then $\iint_S 1\,dS$ = Area of S

Summary: $\boxed{\text{Page 1154}}$

	Equation	Parametric description. ☆	$\vec{t}_u\times\vec{t}_v$	$\lvert\vec{t}_u\times\vec{t}_v\rvert$
Cylinder	$x^2+y^2=a^2$ $0\le z\le h$	$\vec{r}=\langle a\cos u, b\sin u, v\rangle$ $0\le u\le 2\pi$, $0\le v\le h$	$\langle a\cos u, a\sin u, 0\rangle$	a
Cone	$z^2=x^2+y^2$ $0\le z\le h$	$\vec{r}=\langle v\cos u, v\sin u, v\rangle$ $0\le u\le 2\pi$ $0\le v\le h$	$\langle v\cos u, v\sin u, -v\rangle$	$\sqrt{2}\,v$
Sphere	$x^2+y^2+z^2=a^2$	$\vec{r}=\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ $\underset{\varphi}{0\le u\le\pi}$ $\underset{\theta}{0\le v\le 2\pi}$	$\langle a^2\sin^2u\cos v, a^2\sin^2u\sin v, a^2\sin u\cos u\rangle$	$a^2\sin u$
Paraboloid	$z=x^2+y^2$ $0\le z\le h$	$\vec{r}=\langle v\cos u, v\sin u, v^2\rangle$ $0\le u\le 2\pi$ $0\le v\le\sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle$	$v\sqrt{1+4v^2}$
In general	$z=g(x,y)$ or use Type II	$\vec{r}=\langle u, v, g(u,v)\rangle$ $u\Leftrightarrow x$, $v\Leftrightarrow y$	$\langle -g_x, -g_y, 1\rangle$	$\sqrt{1+g_x^2+g_y^2}$

Example P1159 #24 $\iint_S f(x,y,z)\,dS$

$f(x,y,z)=x^2+y^2$, S = is the hemisphere $z\ge 0$ $x^2+y^2+z^2=100$

Answer:

Step 1: Paramertrize the hemisphere

$\vec{r}(u,v) = \langle 10\sin u\cos v, 10\sin u\sin v, 10\cos u\rangle$ $0\le u\le\frac{\pi}{2}$ $0\le v\le 2\pi$.

$\vec{t_u} = \frac{\partial r}{\partial u} = <10\cos u\cos v,\ 10\cos u\sin v,\ -10\sin u>$

$\vec{t_v} = \frac{\partial r}{\partial v} = <-10\sin u\sin v,\ 10\sin u\cos v,\ 0>$

$$\vec{t_u}\times\vec{t_v} = \begin{vmatrix} i & j & k \\ 10\cos u\cos v & 10\cos u\sin v & -10\sin u \\ -10\sin u\sin v & 10\sin u\cos v & 0 \end{vmatrix}$$

$$= 100\sin^2 u\cos v\,\vec{i} + 100\sin^2 u\sin v\,\vec{j} + 100\sin u\cos u\,\vec{k}$$

$$|\vec{t_u}\times\vec{t_v}| = \sqrt{(100\sin^2 u\cos v)^2 + (100\sin^2 u\sin v)^2 + (100\sin u\cos u)^2}$$

$$= \sqrt{100^2\sin^4 u + 100^2\sin^2 u\cos^2 u} = \sqrt{100^2\sin^2 u(\sin^2 u + \cos^2 u)} = 100\sin u$$

$$\iint_S f(x,y,z)\,ds = \iint \left[(10\sin u\cos u)^2 + (10\sin u\sin v)^2\right]\cdot 100\sin u\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[100\sin^2 u(\cos^2 u + \sin^2 u)\right] 100\sin u\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 10000\sin^3 u\,du\,dv \qquad \text{Let } w = \cos u$$

$$= 2\pi\cdot\int_1^0 10^4(1-w^2)(-dw)$$

$$= 2\pi\cdot\int_1^0 10^4\cdot(w^2-1)\,dw$$

$$= 2\pi\cdot 10^4\cdot\left(\frac{w^3}{3} - w\right)\Big|_1^0$$

$$= 2\pi\cdot 10^4\cdot\frac{2}{3}$$

$$= \frac{4\pi}{3}10^4$$

\#

Type II | Surface integral of scalar-valued function on explicitly defined surfaces

Suppose f is a scalar-valued function. S is $z = g(x,y)$ and $(x,y)\in R$. Then

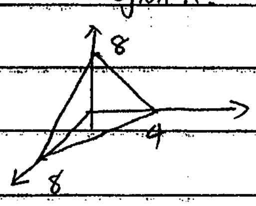
$$\iint_S f(x,y,z)\,ds = \iint_R f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\,dA$$

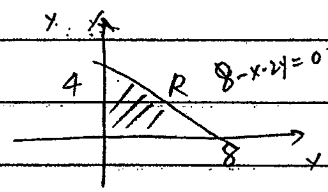
Example: #38 Find $\iint_S f(x,y,z)\,ds$

$f(x,y,z) = e^z$ $\quad$ S: the plane $z = 8 - x - 2y$ in the first Octant

Answer $\quad z_x = -1 \quad z_y = -2 \quad \sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+1+4} = \sqrt{6}$

Region R:





$$\iint_S f\,ds = \int_0^8 \int_0^{4-\frac{x}{2}} e^{8-x-2y}\sqrt{6}\,dy\,dx$$

$$= \int_0^8 -\frac{1}{2}e^{8-x-2y}\sqrt{6}\Big|_0^{4-\frac{x}{2}}\,dx$$

$$= \sqrt{6}\int_0^8 -\frac{1}{2}e^{8-x-8-x} + \frac{1}{2}e^{8-x}\,dx$$

$$= \sqrt{6}\left(-\frac{1}{2}\cdot(8-0) - \frac{1}{2}e^{8-x}\Big|_0^8\right)$$

$$= \left(-4 - \frac{1}{2} + \frac{1}{2}e^8\right)\cdot\sqrt{6}$$

$$= \frac{\sqrt{6}\,(e^8-9)}{2}$$

#

Type III Surface integral of a vector field.

$\vec{F} = \langle f, g, h\rangle$, S is defined parametrically as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v)\in R$

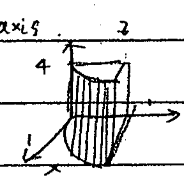
Then $\underbrace{\iint_S \vec{F}\cdot\vec{n}\,ds}_{\text{Flux}} = \iint_R \vec{F}(u,v)\cdot(\pm\,\vec{t}_u\times\vec{t}_v)\,dA$ "$\pm$" depends on orientation

If S is defined $z = s(x,y)$ in region R, then

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_R \vec{F}(x,y)\cdot(\pm\langle -s_x, -s_y, 1\rangle)\,dA$$ "$\pm$" depends on orientation

Example: #48 Find $\iint_S \vec{F}\cdot\vec{n}\,ds$ $\vec{n}$ point in positive y-axis

$\vec{F} = \langle -y, x, 1\rangle$ S: cylinder $y = x^2$ $0\le x\le 1$, $0\le z\le 4$



Answer: $\vec{r}(u,v) = \langle u, u^2, v\rangle$ $0\le u\le 1$ $0\le v\le 4$

$\vec{t}_u = \frac{\partial r}{\partial u} = \langle 1, 2u, 0\rangle$ $\vec{t}_v = \frac{\partial r}{\partial v} = \langle 0, 0, 1\rangle$

$\vec{t}_u\times\vec{t}_v = \langle 2u, -1, 0\rangle$ since $\vec{n}$ is in the positive y-axis direction, we choose $-\vec{t}_u\times\vec{t}_v$ in the formula.

$\vec{F}\cdot(-\vec{t}_u\times\vec{t}_v) = \langle -u^2, u, 1\rangle\cdot\langle -2u, 1, 0\rangle = u + 2u^3$

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^4 u + 2u^3\,dv\,du$$

$$= 4$$

#

Suggested problems Page 1158-1159

Type I: 23 25 27 29

Type II: 32 33 35 37 41

Type III: 43 45 47