§15.4 – 15.5

Topic 1. Green's theorem (§15.4) (2D)

- Green's theorm relates the line integral of a vector field $\vec{F}$ along a closed curve C in 2D and double integral.

Circulation Form: $\oint_C \vec{F}\cdot d\vec{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$ Ⓐ

Flux Form: $\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$ Ⓑ

$\vec{F} = (f, g)$ C is closed and counterclockwise, R is the region bdd by C

- Applications

① If the line integral along a ~~closed~~ closed curve is difficult to calculate, we can change it to double integral. (Ⓐ or Ⓑ) (Note: the general method to calculate line integral is using paramerization of the curve)

② We can calculate the area of a region by changing the double integral to a line integral, the formula is as follows:

$$A = \iint_R 1\,dA = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

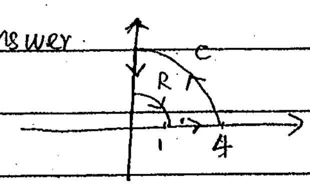
Examples: P1133

#37 $F = \langle \underset{f}{2x+y},\ \underset{g}{x-4y} \rangle$ R: quarter-annulus $\{(r,\theta) : 1 \le r \le 4,\ 0 \le \theta \le \frac{\pi}{2}\}$

C is the boundary of R

Q: Find the circulation and flux of $\vec{F}$ along C.

Answer:



$$\text{Circulation} = \oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

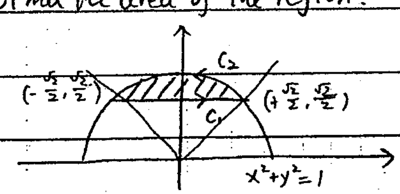
$$= \iint_R (1-1)\,dA = \iint_R 0\,dA = 0$$

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R (2+(-4))\,dA = \iint_R -2\,dA$$

$$= -2 \cdot \text{Area of quarter annulus}$$

$$= -2\cdot\left(\frac{\pi\cdot 4^2}{4} - \frac{\pi\cdot 1^2}{4}\right) = -\frac{15}{2}\pi$$ #

#30 Find the area of the region:



Answer: $C_1: r(t) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + (\sqrt{2}, 0)\,t$

$= \left(-\frac{\sqrt{2}}{2} + \sqrt{2}\,t,\ \frac{\sqrt{2}}{2}\right)$ $0 \le t \le 1$

$C_2: r(t) = \langle \cos t, \sin t \rangle$ $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$

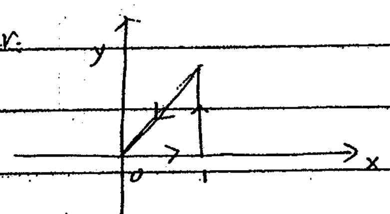
$$\text{Area} = \iint_R 1\,dA = \oint_C x\,dy = \oint_{C_1} x\,dy + \int_{C_2} x\,dy$$

$$= \cancel{\int_0^1 \left(-\tfrac{\sqrt{2}}{2}+\sqrt{2}t\right)\cdot 0 + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos t \cos t\,dt}$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}t + \frac{1}{4}\sin 2t \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{\pi}{4} - \frac{1}{2} \quad \#$$

#4. Find the circulation/flux line integral. $\vec{F} = \langle \overset{f}{e^{x-y}}, \overset{g}{e^{y-x}} \rangle$ C is the boundary of triangle $\{(x,y): 0 \le y \le x,\ 0 \le x \le 1\}$

Answer:



$$\frac{\partial g}{\partial x} = -e^{y-x} \qquad \frac{\partial f}{\partial y} = -e^{x-y}$$

$$\frac{\partial g}{\partial y} = e^{y-x} \qquad \frac{\partial f}{\partial x} = e^{x-y}$$

Circulation line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \iint_R -e^{y-x} + e^{x-y}\,dA$$

$$= \int_0^1 \int_0^x -e^{y-x} + e^{x-y}\,dy\,dx$$

$$= \int_0^1 -e^{y-x} - e^{x-y} \Big|_0^x\,dx$$

$$= \int_0^1 -1 + e^{-x} - 1 + e^{x}\,dx$$

$$= -2x - e^{-x} + e^{x} \Big|_0^1$$

$$= -2 - e^{-1} + e^{1} + \cancel{e^0} - \cancel{e^0} = -2 - e^{-1} + e^{1}$$

Flux line integral

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

$$= \iint_R e^{x-y} + e^{y-x}\,dA = -2 + e + e^{-1} \quad \#$$

Topic 2. Divergence and curl (§15.5)

Def: The divergence of $\vec{F} = \langle f, g, h\rangle$ is $\operatorname{div}\vec{F} = \nabla\cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

If $\nabla\cdot F = 0$, then $\vec{F}$ is called source-free.

Def: The curl of $\vec{F} = \langle f, g, h\rangle$ is $\operatorname{curl}\vec{F} = \nabla\times F = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$

If $\nabla\times F = \vec{0}$, then $\vec{F}$ is called irrotional.

Note: in 2D. $\vec{F} = \langle f, g\rangle$, $\nabla\cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ $\quad \nabla\times F = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\vec{k}$

Special cases:

① Radial field. $\vec{F} = \frac{r}{|r|^p} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}}$. Then $\nabla\cdot F = \frac{3-p}{|r|^p}$ and $\nabla\times F = \vec{0}$

② Rotation field $\vec{F} = \vec{a}\times\vec{r}$ $\vec{a} = \langle a_1, a_2, a_3\rangle$ constant vector and $\vec{r} = \langle x, y, z\rangle$

$\nabla\cdot\vec{F} = 0$ and $|\nabla\times\vec{F}| = 2|\vec{a}|$ $\omega = |\vec{a}|$ is called angular speed

Properties: (More properties P1145 67-72)

Divergence. $\nabla\cdot(\vec{F}+\vec{G}) = \nabla\cdot\vec{F} + \nabla\cdot\vec{G}$

$\nabla\cdot(c\vec{F}) = c(\nabla\cdot\vec{F})$ c is a constant

$\nabla\cdot(u\vec{F}) = \nabla u\cdot\vec{F} + u(\nabla\cdot\vec{F})$ u is a scalar function

Curl: $\nabla\times(\vec{F}+\vec{G}) = \nabla\times\vec{F} + \nabla\times\vec{G}$

$\nabla\times(c\vec{F}) = c(\nabla\times\vec{F})$ c is a constant

- Suppose everything is fine. Then the following statements are equivalent.

① $\vec{F}$ is a conservative vector field

② $\vec{F}$ has potiential function. i.e. $\vec{F} = \nabla\varphi$

③ $\int_C \vec{F}\,d\vec{r} = \varphi(B) - \varphi(A)$ C is any curve with starting point A and ending pt B.

④ $\oint_C \vec{F}\,d\vec{r} = 0$ c is simple, smooth and closed

⑤ $\nabla\times\vec{F} = \vec{0}$

If something bad happens, then they may not be equivalent.

EX: $\vec{F} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$

② ⑤ are true, but ① ③ ④ are false.

EX: P1142-1143

#14 $\vec{F} = \langle \underset{f}{e^{-x+y}}, \underset{g}{e^{-y+z}}, \underset{h}{e^{-z+x}} \rangle$

① $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = -e^{-x+y} - e^{-y+z} - e^{-z+x}$

② $\operatorname{Curl}\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$

$= (0 + e^{-y+z})i + (0 + e^{-z+x})j + (0 + e^{-x+y})\vec{k}$

$= e^{-y+z}\vec{i} + e^{-z+x}\vec{j} + e^{-x+y}\vec{k}$

EX: 24 $\vec{F} = \langle 1, -1, 0\rangle \times \vec{r}$. Find $\nabla\times\vec{F}$ and ω

Answer: $\vec{F} = \langle 1, -1, 0\rangle \times \vec{r} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix} = \underset{f}{-z\,i} - \underset{}{(z)\,j} + \underset{h}{(y+x)\vec{k}}$

$$\nabla \times \vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) i + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) j + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \vec{k}$$

$$= (1+1)\, i + (-1-1)\, j + (0+0)\, \vec{k}$$

$$= 2\vec{i} - 2\vec{j} = \langle 2, -2, 0 \rangle$$

$$w = \frac{1}{2} |\nabla \times \vec{F}| = \frac{1}{2}\sqrt{4+4} = 2\sqrt{2}$$ #

EX: $\varphi = \frac{1}{|r|^2} = \frac{1}{x^2+y^2+z^2}$

① $\vec{F} = \nabla \varphi = \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle = \frac{-2r}{|r|^4}$

② $\nabla \cdot \vec{F} = \nabla \cdot \left(-2 \frac{r}{|r|^4}\right) = -2 \cdot \nabla\left(\frac{r}{|r|^4}\right) = -2 \cdot \frac{3-4}{|r|^4} = \frac{2}{|r|^4}$

③ $\nabla \times \vec{F} = 0$. #

Suggested problems for Q9

Topic1 20, 21 29 31, 32 33 34 35 37 38 P1133

Topic2: 11 13 15 16 17 19 23 25 27 29 31 33 35 38 P1142-1143