Midterm 2 Review

Chapter 13

§13.6.

$z = f(x,y)$, $\vec{u} = \langle u_1, u_2 \rangle$

directional derivative of f in the direction of $\vec{u}$ at point (a,b)

↳ $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \frac{\vec{u}}{|\vec{u}|}$, $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$

Need to know Thm 13.12 and Thm 13.13

§13.7

Given a surface $F(x,y,z) = 0$, its tangent plane at $\langle a,b,c \rangle$

has ▶ normal vector $\nabla F(a,b,c)$

and ▶ equation $\nabla F(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$

or equivalently ▶ $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$.

Specifically, for $z = f(x,y)$ (or $F(x,y,z) = f(x,y) - z = 0$)

The tangent plane at $(a, b, f(a,b))$ has equation

▷ $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$.

▷ $L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$ – the linear approximation to $z = f(x,y)$ for (x,y) near (a,b).

EX. §13.7. #27 Given a surface $z = f(x,y) = -x^2 + 2y^2$; find an equation of the tangent plane at $(3, -1)$ and estimate $f(3.1, -1.04)$.

Answer: $f_x = -2x$, $f_y = 4y$, $f_x(3,-1) = -6$; $f_y = -4$

$f(3,-1) = -9 + 2 = -7$

So the tangent plane has equation

$z = f_x(3,-1)(x-3) + f_y(3,-1)(y+1) + f(3,-1)$

i.e. $z = -6(x-3) - 4(y+1) - 7$

Because $(3.1, -1.04)$ is near $(3, -1)$, we use the linear approx.

$$L(x,y) = -6(x-3) - 4(y+1) - 7$$

to estimate $f(3.1, -1.04)$, i.e.

$$f(3.1, -1.04) \approx L(3.1, -1.04)$$
$$= -6(3.1-3) - 4(-1.04+1) - 7$$
$$= -6 \times 0.1 + 4 \times 0.04 - 7 = \cdots$$

§13.8. Max/Min §13.9 Lagrange multipliers

Need to know the 2nd derivative test (Thm 13.15)

(Read the conditions before the statements 1, 2, 3, 4 of different cases)

Need to know the procedure of using Lagrange multipliers (§13.9)

E.X. §13.9. #15. Find the absolute max and min of $f(x,y) = e^{2xy}$ in the closed disk $R = \{(x,y) \mid x^2+y^2 \le 16\}$.

ⓐ In the interior of R, i.e., the open disk $x^2+y^2 < 16$,

First find critical points by setting

$$\begin{cases} f_x = 2ye^{2xy} = 0 \\ f_y = 2xe^{2xy} = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \Rightarrow \text{Critical point } (0,0).$$

Record $f(0,0) = e^{2\cdot 0\cdot 0} = 1$.

(critical point $(0,0)$ is actually a saddle point (verify this by the 2nd derivative test !))

ⓑ On the boundary $x^2+y^2=16$, we can use Lagrange multipliers.

$$\begin{cases} \text{target } f(x,y) = e^{2xy} \\ \text{constraint } g(x,y) = x^2+y^2-16=0 \end{cases}$$

Set $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$ i.e. $\begin{cases} 2ye^{2xy} = \lambda \cdot 2x & ① \\ 2xe^{2xy} = \lambda \cdot 2y & ② \\ x^2+y^2=16 & ③ \end{cases}$

① and ② imply that none of x, y, λ can be zero.
(e.g. if $\lambda = 0$ then ① and ② $\Rightarrow$ $x = y = 0$ which contradicts ③)

[note: $e^x \neq 0$ for all real numbers x]

So perform $\frac{①}{②}$ to get.

$$\frac{2y e^{2xy}}{2x e^{2xy}} = \frac{\lambda \cdot 2x}{\lambda \cdot 2y} \Leftrightarrow \frac{y}{x} = \frac{x}{y} \Leftrightarrow y^2 = x^2$$

Plug this in ③ $\Rightarrow$ $2x^2 = 16$ $\Rightarrow$ $x^2 = 8$ $\Rightarrow$ $x = \pm 2\sqrt{2}$.

Combined with $y^2 = x^2$, we have <u>four</u> points to consider

(x,y)	$(2\sqrt{2}, 2\sqrt{2})$	$(2\sqrt{2}, -2\sqrt{2})$,	$(-2\sqrt{2}, 2\sqrt{2})$	$(-2\sqrt{2}, -2\sqrt{2})$
$f(x,y) = e^{2xy}$	e^{16}	e^{-16}	e^{-16}	e^{16}

These data combined with $f(0,0)=1$ in part ⓐ $\Rightarrow$

In $R = \{(x,y) \mid x^2+y^2 \leq 1\}$

$f(x,y)$ attains abs max e^{16} at $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$

... abs min e^{-16} at $(2\sqrt{2}, -2\sqrt{2})$ and $(-2\sqrt{2}, 2\sqrt{2})$

Chapter 14.

§14.2.

Need to know Thm 14.2.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

$y=h(x)$ upper curve

$y=g(x)$ lower curve

$$\text{or } = \int_c^d \int_{u(y)}^{v(y)} f(x,y)\,dx\,dy.$$

$x=v(y)$ right curve

$x=u(y)$ left curve

▶ outermost integral should always have constant limits.

§14.3.

Need to know Thm 14.4. polar coordinates

$$\begin{cases} x = r\cos\theta & (x^2+y^2) \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \end{cases}$$

$$\iint_R \underbrace{f(x,y)\,dA}_{} = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}_{}$$

$r=g(\theta)$ inner curve, $r=h(\theta)$ outer curve.

§14.4.

Need to know Thm 14.5 and the discussion above it titled "Finding Limits of Integration"

§14.5. Cylindrical coords

Need to know Thm 14.6

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad (x^2+y^2=r^2) \qquad dV = r\,dz\,dr\,d\theta$$

Table 14.4

Spherical coords

Need to know Thm 14.7

Table 14.5

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} \quad (x^2+y^2+z^2=\rho^2,$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$(0<\rho<\infty,\ 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi)$

§14.6

Know the formulas defining center of mass in 1d, 2d, 3d objects. See the three definitions in this section.

§14.7.

Know Thm 14.8.

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v))\,|J(u,v)|\,dA$$

either dudv or dvdu..

under the change of variables

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

Chapter 15 see recitation notes from the last time

The concept of average of a function on a domain

$$\text{Avg} = \frac{\int_{\text{domain}} \text{function } d\mu}{\int_{\text{domain}} 1 \ d\mu}$$

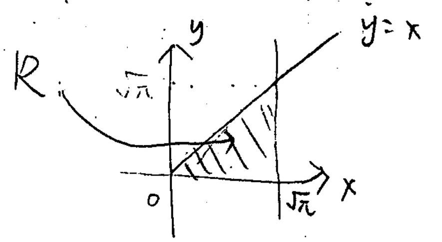
case 1. domain = curve C $\quad \text{Avg} = \dfrac{\int_C f(x,y)\,ds}{\text{length of } C}$ or $\dfrac{\int_C f(x,y,z)\,ds}{\text{length of } C}$

case 2. domain = planar region R $\quad \text{Avg} = \dfrac{\iint_R f(x,y)\,dA}{\text{area of } A}$

case 3. domain = 3D solid D $\quad \text{Avg} = \dfrac{\iiint_D f(x,y,z)\,dV}{\text{volume of } D}$

Ex. Find the average of $f(x,y) = \sin(x^2)$ on the region bounded by lines $y=0$, $x=\sqrt{\pi}$, and $y=x$ in the 1st quadrant.

Answer. Let R be the region in question.



$$\text{Avg} = \frac{\iint_R \sin(x^2)\,dA}{\iint_R 1\,dA} = \frac{\iint_R \sin(x^2)\,dA}{\frac{1}{2}\pi}$$

$\iint_R 1\,dA = \text{area}(A) = \frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi} = \frac{1}{2}\pi$

$\iint_R \sin(x^2)\,dA = \int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2)\,dy\,dx \cdots ①$ $\quad \left\{ \begin{array}{l} \text{upper curve } y=x \\ \text{lower curve } y=0. \end{array} \right.$

or $= \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\,dx\,dy \cdots ②$ $\quad \left\{ \begin{array}{l} \text{right curve } x=\sqrt{\pi} \\ \text{left curve } x=y. \end{array} \right.$

use ① to compute $= \int_0^{\sqrt{\pi}} \left(y\sin(x^2) \Big|_{y=0}^{y=x} \right) dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$

(u-sub $u=x^2$) $\quad = -\frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi}} = 1$

so $\text{Avg} = \dfrac{1}{\frac{1}{2}\pi} = \dfrac{2}{\pi}$.

Ex. §14.5. #18. Evaluate the triple integral

$$I = \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{2} \frac{1}{1+x^2+y^2} \, dz \, dy \, dx$$

Answer. $I = \iiint_D \frac{1}{1+x^2+y^2} \, dV = \iint_R \int_0^2 \frac{1}{1+x^2+y^2} \, dz \, dA$

D is bounded from above by plane $z = 2$.

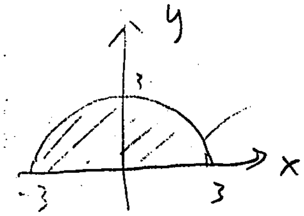
from below . . . . $z = 0$

R is the projection of D onto the xy-plane

$$R = \{ (x,y) \mid -3 \le x \le 3, \ 0 \le y \le \sqrt{9-x^2} \}.$$

$\sqrt{9-x^2} \Rightarrow y^2 + x^2 = 9$

$R \rightsquigarrow$



a semi disk.
circular region.

So we can use the cylindrical coords (r, θ, z).

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad 0 \le r \le 3 \qquad 0 \le \theta \le \pi$$

(note: $x^2+y^2 = r^2$)
$dV = r \, dz \, dr \, d\theta$

So

$$I = \int_0^{\pi} \int_0^{3} \int_0^{2} \frac{1}{1+r^2} \cdot r \, dz \, dr \, d\theta = \int_0^{\pi} \int_0^{3} \left(\frac{r}{1+r^2} \cdot z \Big|_{z=0}^{z=2} \right) dr \, d\theta$$

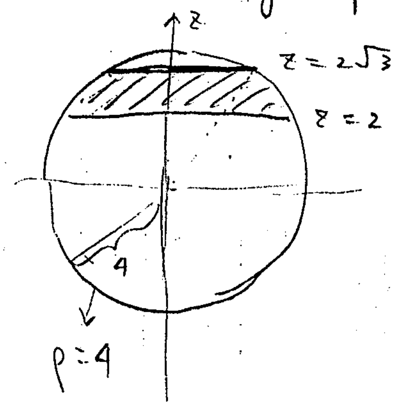
$$= \int_0^{\pi} \int_0^{3} \frac{2r}{1+r^2} \, dr \, d\theta = \int_0^{\pi} \left(\ln(1+r^2) \Big|_{r=0}^{r=3} \right) d\theta$$

u-sub $u = 1+r^2$

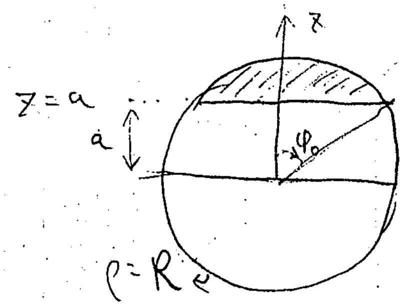
$$= \int_0^{\pi} (\ln(10) - \ln 1) \, d\theta = ((\ln 10) \cdot \theta) \Big|_0^{\pi} = \pi \cdot \ln 10.$$

E.X. §14.5. #51. Find volume of part of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$.

Answer. To visualize the situation we can project the solid D onto the yz-plane, say. The solid D in question is represented by the shaded part:



$$\mathrm{Vol}(D) = \mathrm{Vol}(\text{bigger cap}) - \mathrm{vol}(\text{smaller cap})$$
$$= \mathrm{vol}(\text{bigger}) - \mathrm{vol}(\text{smaller})$$



volume of a general cap.

using cylindrical coords (ρ, φ, θ)

$$\mathrm{Vol}(\text{cap}) = \int_0^{2\pi}\int_0^{\varphi_0}\int_{a\sec\varphi}^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

lower surface $z = a$ $(\rho = a\sec\varphi)$

upper surface $\rho = R$

For our case, $R = 4$,

bigger cap angle $\varphi_b = \frac{\pi}{6}$ (solved from $\cos\varphi_b = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$)

smaller cap angle $\varphi_s = \frac{\pi}{3}$ (solved from $\cos\varphi_b = \frac{2}{4} = \frac{1}{2}$)

Then do the integration.

$$\sec\varphi = \frac{1}{\cos\varphi}$$

$$\mathrm{Vol}(\text{cap}) = \int_0^{2\pi}\int_0^{\varphi_0}\left(\frac{1}{3}\rho^3 \sin\varphi \Big|_{a\sec\varphi}^{R}\right) d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\varphi_0}\left(\frac{1}{3}R^3 \sin\varphi - \frac{1}{3}a^3 \sec^3\varphi \sin\varphi\right) d\varphi \, d\theta$$

$$= \frac{1}{3}R^3 \underbrace{\int_0^{2\pi}\int_0^{\varphi_0} \sin\varphi \, d\varphi \, d\theta}_{\text{easy}} - \frac{1}{3}R^3 \underbrace{\int_0^{2\pi}\int_0^{\varphi_0} \frac{\sin\varphi}{\cos^3\varphi} \, d\varphi \, d\theta}_{\text{u-sub } u = \cos\varphi} = \cdots$$