Math 2153 Nov. 10

§14.7 §15.1–15.2

Topic 1 Recall: change of variable $(x,y,z) \to (u,v,w)$

$$\iiint_D f(x,y,z)\,dV = \iiint_{\widetilde{D}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \underbrace{|J(u,v,w)|}_{\text{absolute value}}\, dV$$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

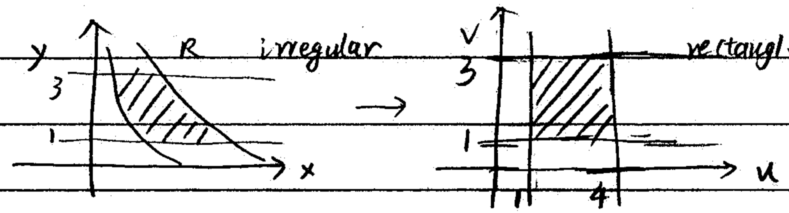
$\widetilde{D}$ is the domain of $\{u,v,w\}$

E.g. §14.7 #35 (Review Problem)

$\iint_R xy\,dA$ R is bounded by $xy=1$ $xy=4$ and $y=1$ and $y=3$

Answer. Note: the purpose of change of variable is transforming R to a more regular region (e.g. rectangle) or making integrand simpler.

Let $u = x\cdot y$ $v = y$ Then $1 \le u \le 4$ and $1 \le v \le 3$



$u = xy$ $v = y$

$$\Rightarrow \begin{cases} x = \frac{u}{y} = \frac{u}{v} \\ y = v \end{cases}$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\iint_R xy\,dA = \iint_{\widetilde{R}} \frac{u}{v}\cdot v\,|J(u,v)|\,dA = \int_1^3\int_0^4 \frac{u}{v}\cdot \cancel{v}\cdot\frac{1}{\cancel{v}}\,du\,dv$$

$$= \int_1^3\int_0^4 \frac{u}{v}\,du\,dv = \frac{15}{2}\ln 3 \quad \#$$

Topic 2. Vector Field.

Def A vector field F is a function that assigns each point (x,y,z) a vector $\langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$

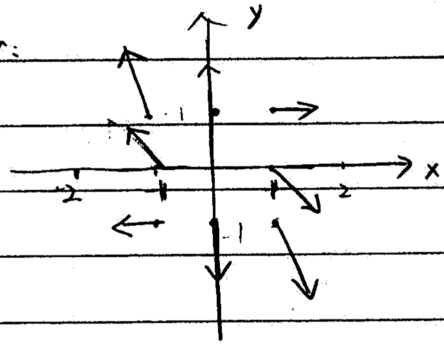
So vector field F can be written as $F(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$

Def: Let φ be a given function, $F = \nabla\varphi$ is called gradient field and φ is called potential function of F. The level curve of potential function $\varphi = C$ is also called equipotential curve.

E.g.: Graph the vector field. §15.1 #13. (Review Problem)

$F = \langle x, y-x \rangle$

Answer:



Points	vectors	points	vectors
$(0,0)$	$\langle 0,0 \rangle$	$(1,1)$	$\langle 1,0 \rangle$
$(1,0)$	$\langle 1,-1 \rangle$	$(1,-1)$	$\langle 1,2 \rangle$
$(0,1)$	$\langle 0,1 \rangle$	$(-1,1)$	$\langle -1,2 \rangle$
$(-1,0)$	$\langle -1,1 \rangle$	$(-1,-1)$	$\langle -1,0 \rangle$
$(0,-1)$	$\langle 0,-1 \rangle$	More points ...	#

E.g. Find gradient field. §15.1 # 35. (Review Problem)

$\varphi = (x^2+y^2+z^2)^{-\frac{1}{2}}$.

Answer. $F = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

$= \langle -x(x^2+y^2+z^2)^{-\frac{3}{2}}, -y(x^2+y^2+z^2)^{-\frac{3}{2}}, -z(x^2+y^2+z^2)^{-\frac{3}{2}} \rangle$

$= -\frac{r}{|r|^3}$ #

§15.2 Line integral.

Parameterization of a given curve.

<1> Circle with center (a,b) and radius r, counterclockwise

$\vec{r}(t) = \langle a + r\cos t, b + r\sin t \rangle \quad 0 \le t \le 2\pi$

<2> Line segment from (a,b) to (c,d)

$\vec{r}(t) = \langle a,b \rangle + t \cdot \langle c-a, d-b \rangle \quad 0 \le t \le 1$

<3> Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ counterclockwise

$\vec{r}(t) = \langle a\cos t, b\sin t \rangle \quad 0 \le t \le 2\pi$

<4> Arbitrary curve $y = f(x)$ $0 \le x \le a$

$\vec{r}(t) = \langle t, f(t) \rangle, \quad 0 \le t \le a$

Type I Scalar line integral.

$\int_C f\,ds = \int_a^b f(x(t), y(t))\,|r'(t)|\,dt$ where $\vec{r}(t) = \langle x(t), y(t) \rangle$ is the parameterization of C

Note: special case. Arc length $= \int_C 1\,ds$

Type II: Line integral of vector field $\vec{F} = \langle f, g \rangle$

$\int_C \vec{F} \cdot \vec{T}\,ds = \int_a^b \vec{F} \cdot \vec{r}'\,dt$ where $\vec{r}(t) = \langle x(t), y(t) \rangle$ is the parameterization of C

$= \int_C \vec{F} \cdot d\vec{r} = \int_C f\,dx + g\,dy$

§15.2 Review Problems

E.g. 29 $\int_C (y-z)\,ds$ C is helix $\vec{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$ $0 \le t \le 2\pi$

Answer: $\int_C (y-z)\,ds = \int_0^{2\pi} (3\sin t - t)\cdot|\vec{r}'(t)|\,dt$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 1\rangle$ $|\vec{r}'(t)| = \sqrt{9(\cos^2 t+\sin^2 t)+1} = \sqrt{10}$

$\int_0^{2\pi} (3\sin t - t)\sqrt{10}\,dt = -3\cos t\Big|_0^{2\pi} - \frac{t^2}{2}\sqrt{10}\Big|_0^{2\pi}$

$= -2\pi^2\sqrt{10}$ #

#35 Line integral of vector field

$\vec{F} = \langle y, x\rangle$, C is a line segment from (1,1) to (5,10)

Answer: Line segment $\vec{r}(t) = \langle 1,1\rangle + t\langle 4,9\rangle = \langle 1+4t, 1+9t\rangle$ $0 \le t \le 1$

$\vec{r}'(t) = \langle 4, 9\rangle$

$\vec{F}(t) = \langle y(t), x(t)\rangle = \langle 1+9t, 1+4t\rangle$

$\vec{F}\cdot\vec{r}' = \langle 1+9t, 1+4t\rangle\cdot\langle 4, 9\rangle = 72t+13$

$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \vec{F}\cdot\vec{r}'\,dt = \int_0^1 72t+13\,dt = 36t^2+13t\Big|_0^1 = 36+13 = 49$

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$\vec{F} = \langle y, x\rangle$ C is $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ $0 \le t \le 2\pi$

(a) Find circulation

Circulation $= \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{r}'(t)\,dt$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t\rangle$

$\vec{F}(t) = \langle y(t), x(t)\rangle = \langle 2\sin t, 2\cos t\rangle$

$\vec{F}\cdot\vec{r}' = \langle 2\sin t, 2\cos t\rangle\cdot\langle -2\sin t, 2\cos t\rangle = -4\sin^2 t + 4\cos^2 t$.

$\int_0^{2\pi} 4\cos^2 t - 4\sin^2 t\,dt = \int_0^{2\pi} 8\cos 2t\,dt = 4\sin 2t\Big|_0^{2\pi} = 0$

(b) Find outflux

Outflux $= \int_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} f(t)\,y'(t) - g(t)\,x'(t)\,dt$

$= \int_0^{2\pi} 2\sin t\cdot 2\cos t - 2\cos t\cdot(-2\sin t)\,dt$

$= \int_0^{2\pi} 4\sin t\cos t\,dt$

$= \int_0^{2\pi} 2\sin 2t\,dt$

$= -\cos 2t\Big|_0^{2\pi}$

$= 0$. #