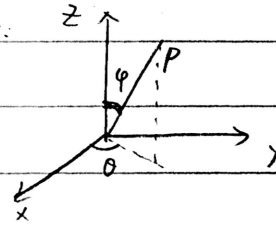
§14.5–14.7

- Review

① Cylindrical: $\iiint_D f\,dv = \int_\alpha^\beta \int_{r_1}^{r_2} \int_{G(r,\theta)}^{H(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$

② Spherical:



$0 \le \varphi \le \pi$ $\qquad x = \rho\sin\varphi\cos\theta$

$0 \le \theta \le 2\pi$ $\qquad y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$$\iiint_D f\,dv = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{G(\theta,\varphi)}^{H(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Equations of some common figures in spherical coordinates

a. Sphere with center $(0,0,0)$ and radius a $\qquad \rho = a$

b. Sphere with center $(0,0,a)$ and radius a $\qquad \rho = 2a\cos\varphi$, $\quad 0 \le \varphi \le \frac{\pi}{2}$.

c. Horizontal plane $(z = a)$. $\quad \rho = a\sec\varphi$, $\quad 0 \le \varphi < \frac{\pi}{2}$ or $\frac{\pi}{2} < \varphi \le \pi$.

d: cone: $\varphi = \varphi_0$

e: cylinder: $(x^2+y^2=a^2)$ $\qquad \rho = a\csc\varphi$

③. Applications of integration: Finding volume, mass, moment and center.

Assume an 3D object: $\quad V = \iiint_D 1\cdot dv$

Mass $= \iiint_D \rho\cdot dv$ where ρ is the density function

Moment $M_x = \iiint_D x\cdot\rho\,dv$

x-coordinate of center $\quad \bar{x} = \dfrac{M_x}{\text{Mass}} = \dfrac{\iiint_D x\rho\,dv}{\iiint_D \rho\,dv}$

④ Change of variables:

Let $x = g(u,v,w)$ and $y = h(u,v,w)$ $z = l(u,v,w)$ be a transformation from xyz-space to uvw-space. Then

$$\iiint_D f(x,y,z)\,dv = \iiint_S f(g(u,v,w), h(u,v,w), l(u,v,w))\,|J(u,v,w)|\,dv$$

where D is the solid in xyz-space. and S is the solid in uvw-space

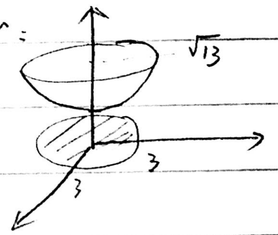
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial x}{\partial u}\begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - \frac{\partial x}{\partial v}\cdot\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + \frac{\partial x}{\partial w}\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

- Strategy for choosing new variables: get hints from either the integrand or the region

HW9 #6

E.g. Find the volume of region bdd by $z=\sqrt{13}$ and $z=\sqrt{4+x^2+y^2}$

Answer:



intersection: $\sqrt{4+x^2+y^2}=\sqrt{13}$

$x^2+y^2=9$

Projection on xy-plane is a disk $x^2+y^2\le 9$

$$V=\iiint_D 1\,dV=\iint_R\int_{\sqrt{4+x^2+y^2}}^{\sqrt{13}}1\,dz\,dA$$

$$=\iint_R\int_{\sqrt{4+r^2}}^{\sqrt{13}}1\,dz\;r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3\left(\sqrt{13}-\sqrt{4+r^2}\right)r\,dr\,d\theta \qquad \text{(u-substitution)}$$

$$=\int_0^{2\pi}\frac{1}{2}\sqrt{13}\,r^2\Big|_0^3-\frac{1}{3}(4+r^2)^{\frac{3}{2}}\Big|_0^3\,d\theta$$

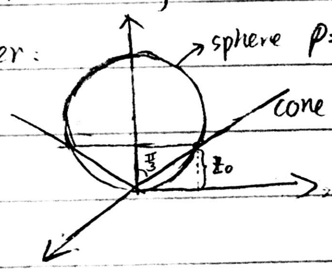
$$=\int_0^{2\pi}\frac{8}{3}+\frac{\sqrt{13}}{6}\,d\theta$$

$$=2\pi\cdot\left(\frac{8}{3}+\frac{\sqrt{13}}{6}\right)$$ #

HW9 #8

E.g. Find the volume of solid inside the cone $\varphi=\frac{\pi}{3}$ and sphere $\rho=4\cos\varphi$ and the solid outside cone and inside sphere.

Answer:



sphere $\rho=4\cos\varphi$ radius $=2$

cone $\varphi=\frac{\pi}{3}$

intersection: $\rho_0=4\cos\frac{\pi}{3}=2$ $\quad z_0=\rho_0\cos\frac{\pi}{3}=2\times\frac{1}{2}=1$

The intersection is a plane $z=z_0=1$. The corresponding equation in spherical coordinates is $\rho=z_0\sec\varphi=\sec\varphi$

① Inside cone and sphere:

$$V=\iiint_D 1\,dV=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\frac{1}{3}\rho^3\sin\varphi\Big|_0^{4\cos\varphi}d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\frac{64}{3}\cos^3\varphi\,\sin\varphi\,d\varphi\,d\theta \qquad u=\cos\varphi \quad du=-\sin\varphi\,d\varphi$$

$$=\int_0^{2\pi}\int\frac{64}{3}u^3(-du)\,d\theta$$

$$\cancel{=\int_0^{2\pi}\frac{64}{3}\cdot\left(-\frac{1}{4}\right)\cdot(\cos\varphi)^4\Big|_0^{\frac{\pi}{3}}d\theta}$$

$1-(\frac{1}{2})^3$

$$=\int_0^{2\pi}\frac{16}{3}\cdot\left(\cos^3 0-\cos^3\frac{\pi}{3}\right)d\theta$$

$$=\int_0^{2\pi}\frac{16}{3}\cdot\frac{7}{8}\,d\theta$$

$$=\frac{14}{3}\cdot\theta\Big|_0^{2\pi}$$

$$=\frac{28}{3}\pi$$

(2) Outside cone and inside sphere

$$V = \iiint_D 1\,dV = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

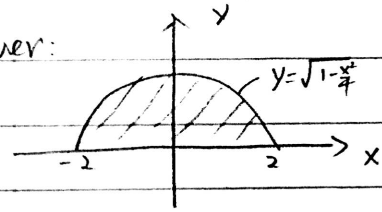
☆ More question. Volume of solid inside cone and sphere and above the intersection?

The intersection is a plane with equation $\rho = \sec\varphi$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\varphi}^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

HW9 #12 Find the center of the plate bounded by $x^2+4y^2 = 4$ and $y \geq 0$ with $\rho(x,y) = 7+y$

Answer:



$$\frac{x^2}{4} + y^2 = 1$$

Since the region is symmetric about y-axis and ρ does not depend on x., $\bar{x} = 0$

(But for the problem that ρ is a function of both x and y, this may not be true!)

Now, we only need to find $\bar{y} = \frac{M_x}{m}$

$$m = \iint_R \rho\, dA = \int_{-2}^{2}\int_0^{\sqrt{1-\frac{x^2}{4}}} (7+y)\,dy\,dx$$

$$= \int_{-2}^{2} 7y + \frac{1}{2}y^2 \Big|_0^{\sqrt{1-\frac{x^2}{4}}} dx$$

$$= \int_{-2}^{2} 7\sqrt{1-\frac{x^2}{4}} + \frac{1}{2}\left(1-\frac{x^2}{4}\right) dx$$

$$= 7\left(\frac{x}{2}\sqrt{1-\frac{x^2}{4}} + \sin^{-1}\frac{x}{2}\right) + \frac{1}{2}x - \frac{1}{24}x^3 \Big|_{-2}^{2}$$

$$= 7\pi + \frac{2}{3}$$

$$M_x = \iint_R y\cdot\rho\, dA$$

$$= \int_{-2}^{2}\int_0^{\sqrt{1-\frac{x^2}{4}}} y(7+y)\,dy\,dx$$

$$= \int_{-2}^{2} \frac{y^2}{2} + \frac{y^3}{3} \Big|_0^{\sqrt{1-\frac{x^2}{4}}} dx$$

$$= \int_{-2}^{2} \frac{1}{2}\left(1-\frac{x^2}{4}\right) + \frac{1}{3}\left(1-\frac{x^2}{4}\right)\sqrt{1-\frac{x^2}{4}}\, dx$$

$$= \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{4}\sin^{-1}\frac{x}{2} - \frac{1}{24}x\sqrt{1-\frac{x^2}{4}}(x^2-10) \Big|_{-2}^{2}$$

$$= \frac{4}{3} + \frac{3}{4}\pi$$

Recall: trig-substitution

$$\boxed{\int \sqrt{1-\frac{x^2}{4}}\, dx}$$

Let $x = 2\sin\theta \quad dx = 2\cos\theta\, d\theta$

$$= \int \sqrt{1-\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$= \int 2\cos^2\theta\, d\theta$$

$$= \int (\cos 2\theta + 1)\, d\theta$$

$$= \frac{1}{2}\sin 2\theta + \theta + C$$

$$= \sin\theta\cos\theta + \theta + C$$

$$= \frac{x}{2}\cdot\sqrt{1-\left(\frac{x}{2}\right)^2} + \sin^{-1}\frac{x}{2} + C$$

$$\boxed{\int \left(1-\frac{x^2}{4}\right)\sqrt{1-\frac{x^2}{4}}\, dx}$$

$x = 2\sin\theta \quad dx = 2\cos\theta$

$$= \int (1-\sin^2\theta)\cos\theta\cdot 2\cos\theta\, d\theta$$

$$= \int \cos^2\theta\cos\theta\cdot 2\cos\theta\, d\theta$$

$$= \int 2\cos^4\theta\, d\theta \quad \text{(left}$$

$$\boxed{\text{By reduction formula}} = \frac{3}{4}\theta + \frac{1}{2}\sin(2\theta) + \frac{1}{16}\sin(4\theta) + C$$

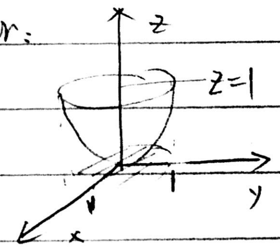
$$= \frac{3}{4}\sin^{-1}\frac{x}{2} - \frac{1}{8}x\sqrt{1-\frac{x^2}{4}}(x^2-10) + C \longleftarrow$$

$$\text{center} = \frac{\frac{4}{3}+\frac{1}{4}\pi}{7\pi+\frac{2}{3}}$$ #

HW9 #13

E.g. Find the center of solid bounded by $z=x^2+y^2$ and $z=1$ with $\rho=1$

Answer:



intersection $x^2+y^2=1$

Projection on xy-plane is a disk.

Since the solid is symmetric about z-axis, and the density function is a constant function,

$\bar{x}=\bar{y}=0$. We only need to find $\bar{z}$.

$$\bar{z}=\frac{M_{xy}}{m}.$$

$$m=\iiint_D \rho\, dv = \iiint_D 1\, dv = \iint_R \int_{\sqrt{x^2+y^2}}^{1} 1\, dz\, dA$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 1\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1-r)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r-r^2\, dr\, d\theta$$

$$= \frac{\pi}{3}$$

$$M_{xy}=\iiint_D z\rho\, dv = \iiint z\, dv = \iint_R \int_{\sqrt{x^2+y^2}} z\, dz\, dA$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}z^2\Big|_r^1\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{2}-\frac{1}{2}r^2\right)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}r-\frac{1}{2}r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}r^2-\frac{1}{8}r^4\Big|_0^1\, d\theta$$

$$= \left(\frac{1}{4}-\frac{1}{8}\right)\cdot 2\pi$$

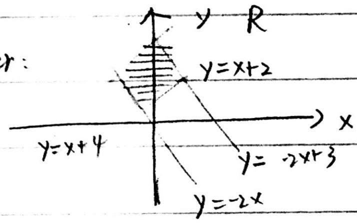
$$= \frac{1}{4}\pi$$

$$\bar{z}=\frac{\frac{1}{4}\pi}{\frac{1}{3}\pi}=\frac{3}{4}$$

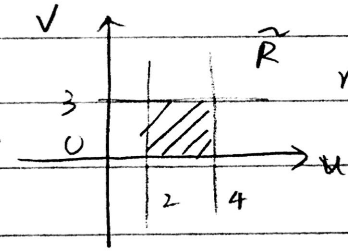
HW10 #6

E.g $\iint_R \left(\frac{y-x}{y+2x+1}\right)^2 dA$ R is bounded by $y-x=2$ $y-x=4$ $y+2x=0$ and $y+2x=3$

Answer:



$$\xrightarrow[v=y+2x]{u=y-x}$$



rectangle area

$$\begin{cases} u = y-x \\ v = y+2x \end{cases} \Rightarrow \begin{cases} x = \frac{v-u}{3} \\ y = \frac{2u+v}{3} \end{cases}$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$$\iint_R f\,dA = \iint_{\tilde{R}} f \cdot |J(u,v)|\, dA.$$

$$= \int_0^3 \int_2^4 \left(\frac{u}{v+1}\right)^2 \left|-\frac{1}{3}\right| du\,dv$$

$$= \frac{1}{3} \int_0^3 \int_2^4 \frac{u^2}{(v+1)^2}\, du\,dv$$

$$= \frac{1}{3} \int_0^3 \left(\frac{1}{v+1}\right)^2 \frac{1}{3} u^3 \Big|_2^4 \, dv$$

$$= \frac{56}{9} \int_0^3 (v+1)^{-2}\, dv$$

$$= \frac{56}{9} (-1)(v+1)^{-1} \Big|_0^3$$

$$= \frac{14}{3}$$

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