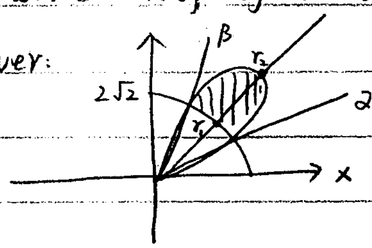
Math 2153 §14.3 – 14.5

Topic 1: Double integrals in Polar Coordinates

$$\iint_D f\, dA = \int_\alpha^\beta \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\cdot r\, dr\, d\theta$$

HW8 #14 Ex: Find the area of region inside rose $r=4\sin 2\theta$ and outside circle $r=2\sqrt{2}$ in 1st quadrant

Answer:



intersections

$4\sin 2\theta = 2\sqrt{2}$

$\sin 2\theta = \frac{\sqrt{2}}{2}$

$2\theta = \frac{\pi}{4}$ $\quad \theta = \frac{\pi}{8} = \alpha$

or $2\theta = \frac{3\pi}{4}$ $\quad \theta = \frac{3\pi}{8} = \beta$

$$\text{Area} = \iint_D 1\, dA = \int_\alpha^\beta \int_{r_1}^{r_2} r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \int_{2\sqrt{2}}^{4\sin 2\theta} r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{r^2}{2}\Big|_{2\sqrt{2}}^{4\sin 2\theta} d\theta$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 8\sin^2 2\theta - 4\, d\theta$$

Double angle formula!

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 8\left(\frac{1-\cos 4\theta}{2}\right) - 4\, d\theta$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} -4\cos 4\theta\, d\theta$$

$$= -\sin 4\theta \Big|_{\frac{\pi}{8}}^{\frac{3\pi}{8}} = -\sin\left(\tfrac{3\pi}{2}\right) + \sin\tfrac{\pi}{2} = -(-1)+1 = 2$$

Topic 2 Triple integrals in Cartesian Coordinates

$$\iiint_D f\, dV$$

Step 1: write it as $\iint_R \int_{G(x,y)}^{H(x,y)} f\, dz\, dA$ $\quad$ R is projection of D on xy-plane

or $\iint_R \int_{G(x,z)}^{H(x,z)} f\, dy\, dA$. $\quad$ R is projection of D on xz plane

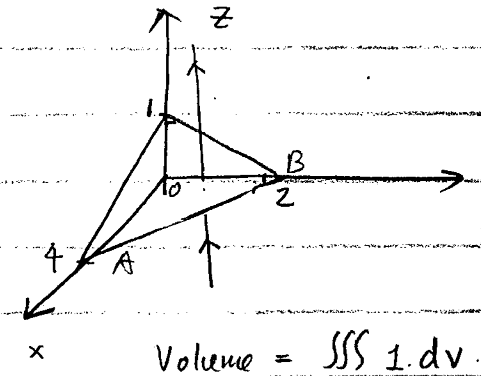
or $\iint_R \int_{G(y,z)}^{H(y,z)} f\, dx\, dA$, $\quad$ R is projection of D on yz-plane.

Step 2: after evaluation of inner-most integral, deal with the double integral $\iint_R \boxed{\cdots}\, dA$

HW8 #20

EX: Find volume of given solid in first octant bounded by $2x+4y+8z=8$ and coordinate planes

Answer: Graphing!



Draw a line parallel to z-direction

$G(x,y)$: the surface where the line enter the solid D

$H(x,y)$: the surface where the line exit the solid D

$$z = \frac{8-2x-4y}{8}$$

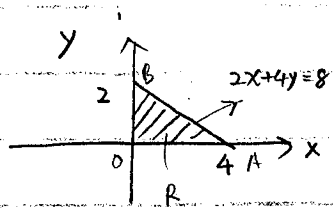
$$\text{Volume} = \iiint_D 1\, dv$$

Step 1: $$= \iint_R \int_{G(x,y)}^{H(x,y)} 1\, dz\, dA \qquad R \text{ is triangular region } OAB$$

$$= \iint_R \int_0^{\frac{8-2x-4y}{8}} 1.\, dz\, dA = \iint_R \frac{8-2x-4y}{8}\, dA$$

Step 2: $$\text{Vol} = \iint_R \frac{8-2x-4y}{8}\, dA$$

$$= \int_0^4 \int_0^{\frac{8-2x}{4}} 1 - \frac{x}{4} - \frac{y}{2}\, dy\, dx$$



$$\text{or} = \int_0^2 \int_0^{\frac{8-4y}{2}} 1 - \frac{x}{4} - \frac{y}{2}\, dx\, dy$$

$$= \frac{4}{3}$$

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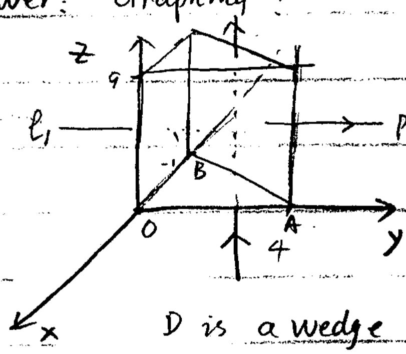
HW8 #22

EX: Change of order. (Similar HW8 #24)

$$\underbrace{\int_0^9 \int_{-1}^0}_{R} \underbrace{\int_0^{4x+4} dy\, dx\, dz}_{\cancel{xz\text{-plane}}} \longrightarrow \iiint dz\, dx\, dy.$$

Answer: Graphing!



D is a wedge.

① l_1 enters solid through $y=0$

exits solid through $y=4x+4$ plane // z-axis

② Projection on xz-plane is a rectangular region

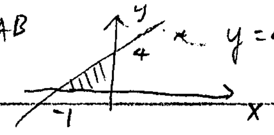
$-1 \le x \le 0$

$0 \le z \le 9$

Now: change order: innermost is z direction, draw a line l_2 parallel to z

projection on xy plane is a triangular region OAB

① ℓ_z enters solid through $z=0$, exists solid through $z=9$

② projection on xy-plane OAB  $y=4x+4$ or $x=\frac{y-4}{4}$

$$= \iint_R \int_0^9 dz\, dA$$

$$= \int_0^4 \int_0^{\frac{y-4}{4}} \int_0^9 dz\, dx\, dy$$

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Topic 3. Triple integral in cylindrical coordinates

$$\iiint_R dv = \iint_R \int_{G(r,\theta)}^{H(r,\theta)} f(r\cos\theta, r\sin\theta, z)\cdot dz\, r\,dr\,d\theta.$$

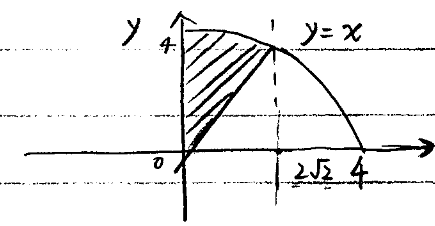
R is projection on xy-plane.

~~HW9 #4~~

EX: Cartesian $\rightarrow$ Cylindrical.

$$\int_0^3 \int_0^{2\sqrt{2}} \int_x^{\sqrt{16-x^2}} e^{-x^2-y^2}\, dy\, dx\, dz$$

Answer: Graphing! Only needs 2D graph on xy-plane for this problem.



lower bound: $y = x$

upper bound $y=\sqrt{16-x^2} \iff x^2+y^2=16$

$$\int_0^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^4 e^{-r^2}\, r\,dr\,d\theta\, dz$$

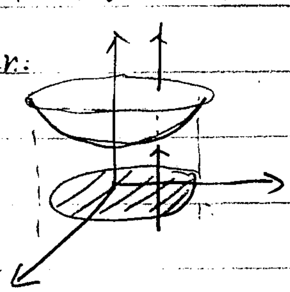
$$= \int_0^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\tfrac{1}{2}e^{-r^2}\Big|_0^4\, d\theta\, dz$$

$$= \int_0^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\tfrac{1}{2}\left[e^{-16}-e^0\right] d\theta\, dz$$

$$= \frac{3\pi}{8}\left(1-e^{-16}\right)$$

HW9 # 6. Find volume bounded by $z=\sqrt{13}$ and $z=\sqrt{4+x^2+y^2}$

Answer:



Intersection: $\sqrt{13}=\sqrt{4+x^2+y^2} \iff x^2+y^2=9 \quad r=3$

$$\text{Volume} = \iiint_D f\, dv$$

$$= \iint_R \int_{\sqrt{4+r^2}}^{\sqrt{13}} dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^3 \int_{\sqrt{4+r^2}}^{\sqrt{13}} 1 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 (\sqrt{13} - \sqrt{4+r^2}) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \sqrt{13} \, \frac{r^2}{2} - \frac{1}{3}(4+r^2)^{\frac{3}{2}} \Big|_0^3 \, d\theta$$

$$= \int_0^{2\pi} \frac{9}{2}\sqrt{13} - \frac{1}{3} \cdot 13\sqrt{13} - (0 - \frac{1}{3} \cdot 2^3) \, d\theta$$

$$= 2\pi \left(\frac{1}{6}\sqrt{13} + \frac{8}{3} \right)$$

$$= \frac{1}{3}(16 + \sqrt{13})\pi$$

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