Math 2153

△ § 13.9 Lagrange Multipliers

Goal: Find extrema of objective function $f(x,y,z)$ under constraint $g(x,y,z)=0$

Procedure: set up $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

Solve the system of equation and evaluate f at the solution.

EX: (Quiz 7).

Find max and min of $f(x,y,z) = 10x - 6y + 8z$ subject to $x^2+y^2+z^2=50$

Answer: Objective function: $f(x,y,z) = 10x-6y+8z$

Constraint: $g(x,y,z) = x^2+y^2+z^2-50=0$

set up: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0. \end{cases}$

$$\Leftrightarrow \begin{cases} 10 = \lambda \cdot 2x & ① \Rightarrow x = \frac{5}{\lambda} \\ -6 = \lambda \cdot 2y & ② \Rightarrow y = -\frac{3}{\lambda} \\ 8 = \lambda \cdot 2z & ③ \Rightarrow z = \frac{4}{\lambda} \\ x^2+y^2+z^2=50 & ④ \end{cases} \quad \text{plug in ④}$$

$$\left(\frac{5}{\lambda}\right)^2 + \left(-\frac{3}{\lambda}\right)^2 + \left(\frac{4}{\lambda}\right)^2 = 50. \Leftrightarrow \frac{50}{\lambda^2} = 50 \Leftrightarrow \lambda = \pm 1$$

when $\lambda = 1$, $x=5$, $y=-3$, $z=4$ and $f(5,-3,4) = 100$ max

when $\lambda=-1$ $x=-5$ $y=3$ $z=-4$ and $f(-5,3,4) = -100$ min #

△ § 14.1 – 14.2

Topic 1: How to choose the order to integrate? Answer: start from the variable that is easy to integrate

EX: $\int_0^{\frac{\pi}{3}} \int_0^1 y\cos(xy)\,dx\,dy$ or $\int_0^1 \int_0^{\frac{\pi}{3}} y\cos(xy)\,dy\,dx$?

Answer: If we start from y, then we need integration by parts! So start from x.

$$= \int_0^{\frac{\pi}{3}} \sin(xy)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^{\frac{\pi}{3}} \sin(y) - \sin 0 \, dy$$

$$= \int_0^{\frac{\pi}{3}} \sin y \, dy = (-\cos y)\Big|_0^{\frac{\pi}{3}}$$

$$= -\cos\frac{\pi}{3} + \cos 0 = 1 - \frac{1}{2} = \frac{1}{2}$$

HW 7

EX: $\int_0^6 \int_1^2 \frac{x}{(4+xy)^2}\,dy\,dx$ or $\int_1^2 \int_0^6 \frac{x}{(4+xy)^2}\,dx\,dy$?

Answer: If we start from x, both numerator and denominator have x. So start from y.

$$= \int_0^6 \int_1^2 \frac{x}{(4+xy)^2}\, dy\, dx$$

$$= \int_0^6 \int_1^2 x(4+xy)^{-2}\, dy\, dx \qquad \text{Let } 4+xy = u \qquad x\,dy = du$$

$$= \int_0^6 \left(-\frac{1}{4+xy}\right)\Bigg|_{y=1}^{y=2} dx \qquad \int x(4+xy)^{-2} dy = \int u^{-2} du = -u^{-1}$$

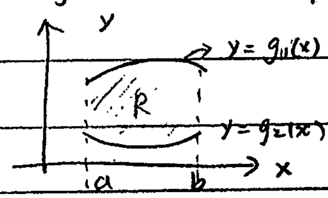
$$= \int_0^6 \left(-\frac{1}{4+2x} + \frac{1}{4+x}\right) dx \qquad = -(4+xy)^{-1}$$

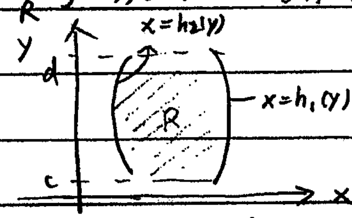
$$= -\frac{1}{2}\ln(4+2x)\Big|_0^6 + \ln(4+x)\Big|_0^6$$

$$= -\frac{1}{2}\ln 16 + \frac{1}{2}\ln 4 + \ln 10 - \ln 4 = \ln 5 - 2\ln 2 \qquad \#$$

Topic 2 Double integral on arbitrary region R: $\iint_R f(x,y)\, dA \qquad dA = dx\,dy$ or $dA = dy\,dx$

Recall



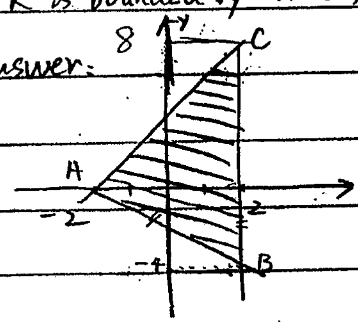


$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y)\, dy\, dx \qquad \iint_R f(x,y)\, dA = \int_c^d \int_{h_2(y)}^{h_1(y)} f(x,y)\, dx\, dy$$

Graphing is very helpful!

HW8 EX R is bounded by $x=2$, $y=2x+4$, $y=-x-2$. Find $\iint_R y^2\, dA$

Answer:



intersections

A: $\begin{cases} y = 2x+4 \\ y = -x-2 \end{cases} \Rightarrow x = -2 \quad y = 0 \quad (-2, 0).$

$3x = -6$

B: $(2, -4)$

C: $(2, 8)$

Method I: $\int_{-2}^{2} \int_{-x-2}^{2x+4} y^2\, dy\, dx$.

$$= \int_{-2}^{2} \frac{1}{3} y^3 \Big|_{(-x-2)}^{(2x+4)} dx$$

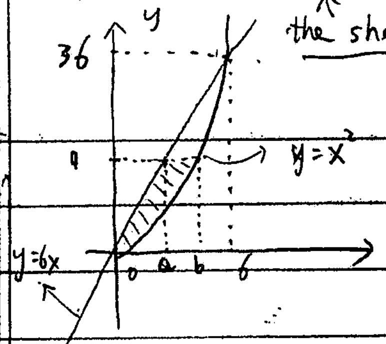
$$= \int_{-2}^{2} \frac{1}{3}(2x+4)^3 - \frac{1}{3}(-x-2)^3\, dx$$

$$= \cdots = 192$$

Method II: $\int_{-4}^{0} \int_{-y-2}^{2} y^2\, dx\, dy + \int_0^8 \int_{\frac{y-4}{2}}^{2} y^2\, dx\, dy$

EX Change the order of integration. $\int_0^1 \int_{\frac{y}{6}}^{\sqrt{y}} f(x,y)\, dx\, dy$

Graphing the region of integration.



the shadow part

left $x=\frac{y}{6}$, right $x=\sqrt{y}$
$\Updownarrow$ $\Updownarrow$
$y=x^2$ $y=6x$
upper $y=1$, lower $y=0$

intersection of $y=x^2$ and $y=6x$:
$x^2=6x \iff x(x-6)=0$
$\iff x=0$ or $x=6$.

plug $y=1$ in $x=\frac{y}{6} \Rightarrow a=\frac{1}{6}$, plug $y=1$ in $x=\sqrt{y} \Rightarrow b=1$

So the original integral equals

$$\int_0^{\frac{1}{6}}\int_{x^2}^{6x} f(x,y)\,dy\,dx + \int_{\frac{1}{6}}^{1}\int_{x^2}^{1} f(x,y)\,dy\,dx$$ #

Topic 3 Applications

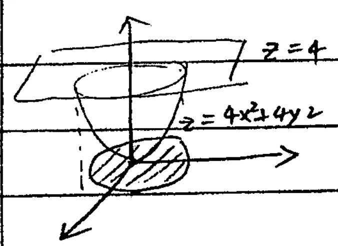
① Average value of a function $\bar{f} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA$

② Area of region R Area of $R = \iint_R 1\,dA$ =

③ Volume between two surfaces. Volume $= \iint_{\boxed{\text{projection on } xy \text{ plane}}} \underset{\text{upper}}{f_1} - \underset{\text{lower}}{f_2}\,dA$.

Ex. Find volume of bounded by two surfaces $z=16$ and $z=4x^2+4y^2$.

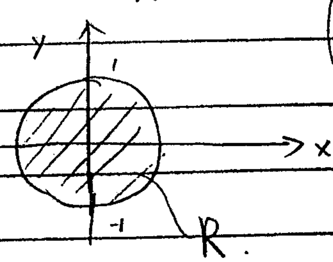
Answer: Needs to find R and also needs to find which surface is on top.



The intersection of two surfaces determine the projection (shadow) of the object on xy-plane

Intersection: $4x^2+4y^2=4 \iff x^2+y^2=1$.

$$\iint_R 4-(4x^2+4y^2)\,dA \qquad R \text{ is the disk } \{(x,y)\mid x^2+y^2\le 1\}$$



$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\underset{\text{upper}}{4} - \underset{\text{lower}}{(4x^2+4y^2)}]\,dy\,dx$$

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