Topic 1. Tangent planes and linear approximation (§13.7)

A surface in $\mathbb{R}^3$ has two forms: Explicit $z = f(x,y)$ → $F(x,y,z) = f(x,y) - z = 0$

Implicit $F(x,y,z) = 0$

- Find a tangent plane at point (a,b,c)

Recall: to get the equation of a plane, we need normal vector $\vec{n} = \langle A, B, C\rangle$ and a point (x_0, y_0, z_0)

Then the equation is $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

The tangent plane at (a,b,c) of a surface has normal vector

$$\vec{n} = \nabla F(a,b,c) = \langle F_x(a,b,c),\ F_y(a,b,c),\ F_z(a,b,c)\rangle \quad \text{"Implicit"}$$

$$= \langle f_x(a,b),\ f_y(a,b),\ -1\rangle \quad \text{"explicit"}$$

So the equation of tangent plane is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0 \quad \text{"implicit"}$$

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0 \quad \text{"explicit"}$$

- The linear approximation $L(x,y)$ at (a,b) of the surface $z = f(x,y)$ is the tangent plane

i.e. $L(x,y) = z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

- We can estimate the z value at a point (x_0, y_0) near (a,b) by

$$z \approx L(x_0,y_0) = f_x(a,b)(x_0-a) + f_y(a,b)(y_0-b) + f(a,b)$$

- Estimate the change ~~of~~ change of a function. (Read pages 966 – 967)

Example: Page 969

\# 19 $z = e^{xy}$

(a) Find the tangent plane at $(0,1,1)$

Answer: The normal vector $\vec{n} = \langle f_x(0,1), f_y(0,1), -1\rangle$

$f_x(x,y) = y e^{xy}$ $\quad f_x(0,1) = 1$ $\quad \Rightarrow \quad \vec{n} = \langle 1, 0, -1\rangle$

$f_y(x,y) = x e^{xy}$ $\quad f_y(0,1) = 0$

The tangent plane is $1\cdot(x-0) + 0\cdot(y-1) - 1(z-1) = 0$

$$\Leftrightarrow \quad x - z + 1 = 0$$

$$\Leftrightarrow \quad z = x + 1$$

(b) Find the pt(s) at which the tangent plane is horizontal. (// xoy-plane)

Answer: A plane is horizontal $\Leftrightarrow$ the normal vector has the form $\vec{n} = \langle 0, 0, c\rangle$

So we are looking for the pt(s) at which $\vec{n} = \langle 0, 0, c \rangle$

$\vec{n} = \langle f_x, f_y, -1 \rangle = \langle y e^{xy}, x e^{xy}, -1 \rangle$

Let $\begin{cases} y e^{xy} = 0 \\ x e^{xy} = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$.

Plug $x=0$ $y=0$ in $z = e^{xy}$ to get z-coordinate $z = e^0 = 1$

At point $(0, 0, 1)$, the tangent plane is horizontal.

(c) Find the linear approximation to $z = e^{xy}$ at $(0, 1, 1)$

Answer: $L(x,y) = x + 1$ (just the tangent plane and we got it in (c))

(d) Use the linear approximation to estimate z at $(x, y) = (0.01, 0.9)$

$z \approx L(0.01, 0.9) = 0.01 + 1 = 1.01$ #

(*) Topic 2 Local extrama (§13.8) (Read pages 974 – 975)

Review: Given $z = f(x,y)$

- 2nd derivative Test: Suppose (a,b) is a critical point, i.e. $f_x(a,b) = 0$ $f_y(a,b) = 0$

  Then we calculate $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2$

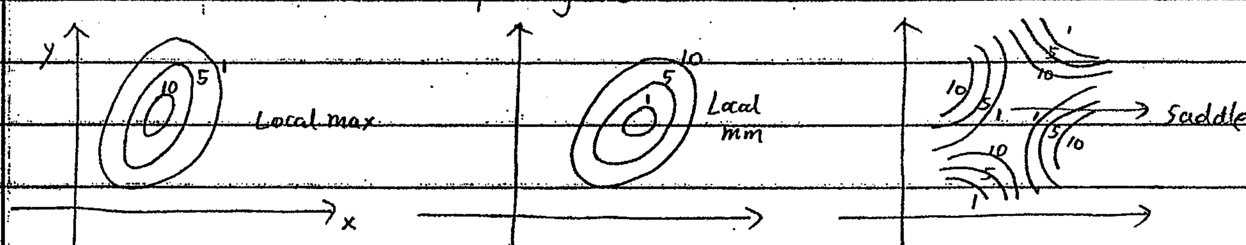
(1) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, the f has a local max at (a,b)

(2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b)

(3) If $D(a,b) < 0$, then f has a saddle pt at (a,b)

(4) If $D(a,b) = 0$, then the test is inconclusive. Use other methods! (ex: graphing)

- The local extrema and corresponding level curves



- Example. Page 982 #33

$f(x,y) = y e^x - e^y$ Analyze the critical pts

Answer:

Step 1: Find the critical points. Solve for x and y from $f_x = 0$, $f_y = 0$.

$f_x = y e^x$ $f_y = e^x - e^y$

$$\begin{cases} ye^x = 0 & ① \\ e^x - e^y = 0 & ② \end{cases}$$

① $\Rightarrow$ $y=0$ plug in ② and find $x=0$

The critical pt is $(0,0)$

Step 2: Use 2nd derivative test to analyze the critical point.

$f_{xx} = ye^x$ $f_{xy} = e^x$ $f_{yy} = -e^y$

$D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2$

$= 0 \cdot e^0 - (-e^0) = 0 - 1 = -1 < 0$

By 2nd derivative test, $(0,0)$ is a saddle pt. #

Topic 3: Absolute extrema in closed region. (§13.8)

Procedure: Step 1: Evaluate $f(x,y)$ at critical pts (Local min/max)

★ Step 2: Evaluate $f(x,y)$ on the boundary of the closed region.

Techniques: for arbitrary region: substition !

for disk: substitution or polar coordinate transformation!

Step 3: Absolute max ← largest f in step 1 and 2

Absolute min ← smallest f in step 1 and 2

Example: Page 982

\# 47 $f(x,y) = 2x^2 - 4x + 3y^2 + 2$ $R: \{(x,y): (x-1)^2 + y^2 \le 1\}$

Find absolute max and min.

Answer: Step 1: Evaluate f at critical pts

$f_x = 4x - 4$ $f_y = 6y$

Let $f_x = f_y = 0$ $\Rightarrow$ $x=1$ $y=0$ $(1,0)$ critical pt

$f(1,0) = 2 - 4 + 0 + 2 = 0$

Step 2: Evaluate f on $(x-1)^2 + y^2 = 1$ (boundary of R)

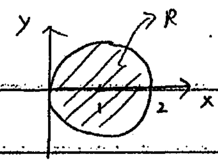
Method 1: Substitution $y^2 = 1 - (x-1)^2$

$f(x,y) = 2x^2 - 4x + 3[1 - (x-1)^2] + 2$

$= \cancel{2x^2} - \cancel{4x} + 3 - \cancel{3x^2} - 3 + \cancel{6}x + 2$

$= -x^2 + 2x + 2$

$= -(x-1)^2 + 3$



So $0 \le x \le 2$

$-1 \le x-1 \le 1$

$0 \le (x-1)^2 \le 1$

$-1 \le -(x-1)^2 \le 0$

$2 \le -(x-1)^2+3 \le 3$

So on the boundary, $2 \le f \le 3$

Step 3: Absolute max $= 3$

Absolute min $= 0$

Method 2: (Method in your book) in step 2

Let $\begin{cases} x = 1+\cos\theta \\ y = \sin\theta \end{cases}$

Then $f = 2(1+\cos\theta)^2 - 4(1+\cos\theta) + 3\sin^2\theta + 2$

$= 2\cos^2\theta + 3\sin^2\theta$

$= 2 + \sin^2\theta$

Since $-1 \le \sin\theta \le 1$, $0 \le \sin^2\theta \le 1$ and $2 \le f \le 3$ #

Topic 1: Langrange Multipliers (§13.9)

- Review: Objective function: $z = f(x,y)$ or $W = f(x,y,z)$

  Constraint $g(x,y)=0$ or $g(x,y,z)=0$

  To maximize / minimize f under $g=0$

  step 1: Solve for $x, y, (z), \lambda$ from $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

  Step 2: Evaluate f at the points you found in step 1 to find max and min.

- Example:

  Find the point on the plane $5x+4y+z=12$ that is nearest to $(2,0,1)$

  Answer: $\boxed{\text{Method 2}}$: Key point: Find objective function and constraint

  $\boxed{\text{Method 1}}$: Substitution + 2nd derivative test

  See example 9 on page 981

  The distance $d(x,y,z) = \sqrt{(x-2)^2+y^2+(z-1)^2}$

  To minimize $d(x,y,z) \Leftrightarrow$ to minimize $d^2(x,y,z)$

  Objective function: $f(x,y,z) = d^2(x,y,z) = (x-2)^2+y^2+(z-1)^2$

  Constraint: $g(x,y,z) = 5x+4y+z-12=0$

  $\nabla f = \langle 2(x-2),\ 2y,\ 2(z-1) \rangle$

  $\nabla g = \langle 5, 4, 1 \rangle$

  Solve for x, y, z, λ from
  $$\begin{cases} 2(x-2) = 5\lambda & ① \\ 2y = 4\lambda & ② \\ 2(z-1) = \lambda & ③ \\ 5x+4y+z=12 & ④ \end{cases}$$

  $① \Rightarrow x = \frac{5\lambda}{2}+2 \quad ② \Rightarrow y = 2\lambda \quad ③ \Rightarrow z = \frac{\lambda}{2}+1$

  Plug in ④. $5\cdot\left(\frac{5\lambda}{2}+2\right) + 4\cdot 2\lambda + \frac{\lambda}{2}+1 = 12 \Rightarrow \lambda = \frac{1}{21}$

  So $x = \frac{5}{2}\cdot\frac{1}{21}+2 = \frac{5}{42}+2 = \frac{89}{42}$

  $y = 2\lambda = \frac{2}{21}$

  $z = \frac{\lambda}{2}+1 = \frac{1}{42}+1 = \frac{43}{42}$

  The point is $\left(\frac{89}{42}, \frac{2}{21}, \frac{43}{42}\right)$. Since the objective function f has the absolute min at the nearest point, and the absolute min is a local min, the nearest point must also satisfy the multiplier equations ①~④, which have only one solution. So the nearest point coincide with the solution $\left(\frac{89}{42}, \frac{2}{21}, \frac{43}{42}\right)$.

Sometimes, the difficult part is solving the equations $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

Example

$$\begin{cases} 6y e^{6xy} = \lambda \cdot 2x & ① \\ 6x e^{6xy} = \lambda \cdot 2y & ② \\ x^2 + y^2 - 38 = 0 & ③ \end{cases}$$

target function $f(x,y) = e^{6xy}$

constraint $x^2+y^2 = 38$

$\frac{①}{②}$: $\frac{y}{x} = \frac{x}{y} \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$

Plug in ③ $x^2+y^2-38 = x^2+x^2-38 = 0 \Rightarrow x^2 = 19 \quad x = \pm\sqrt{19}$.

Hence $y = \mp\sqrt{19}$

Example

$$\begin{cases} 6 = \lambda \cdot 6x & ① \\ 0 = \lambda \cdot 2y & ② \\ 2z = \lambda \cdot 4z & ③ \\ 3x^2+y^2+2z^2-75 = 0 & ④ \end{cases}$$

target function $f(x,y,z) = 6x + z^2$

constraint $3x^2+y^2+2z^2 = 75$

a. From ② $\Rightarrow \lambda = 0$ or $y = 0$ However, if we plug $\lambda = 0$ in ①, we get $6 = 0$ Nonsense!

So $y = 0$.

b. From ③ $z(4\lambda - 2) = 0 \Rightarrow z = 0$ or $\lambda = \frac{1}{2}$

Case 1: $z = 0$,

Plug $z=0, y=0$ in ④ $\Rightarrow x = \pm 5$

Case 2: $\lambda = \frac{1}{2}$ but $z \neq 0$

Plug $\lambda = \frac{1}{2}$ in ① $\Rightarrow x = 2$

Plug $x = 2, y = 0$ in ④ $\Rightarrow z = \pm\sqrt{\frac{63}{2}}$

All solutions: $(2, 0, \sqrt{\frac{63}{2}})$, $(2, 0, -\sqrt{\frac{63}{2}})$, $(5, 0, 0)$ $(-5, 0, 0)$ #

Topic 2. Double integrals (§14.1 - §14.2)

Type I: Rectangular regions

Example: Page 1005 #26

$\iint_A y\cos(xy)\, dA$, $A = \{(x,y): 0 \le x \le 1, 0 \le y \le \frac{\pi}{3}\}$

How do we choose the order of integral? Principle: to avoid complicated calculation

If we first integrate $y\cos(xy)$ wrt y, then we need integration by parts, Complicated!

So we integrate $y\cos(xy)$ wrt x first. (y is treated as a constant)

$\int_0^{\frac{\pi}{3}} \int_0^1 y\cos(xy)\,dx\,dy$ "evaluate inner integral wrt x"

$= \int_0^{\frac{\pi}{3}} \sin(xy)\Big|_{x=0}^{x=1} dy$ "treat y as a constant"

$= \int_0^{\frac{\pi}{3}} (\sin y - \sin 0)\,dy$

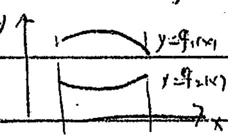
$= \int_0^{\frac{\pi}{3}} \sin y\,dy$

$= -\cos y \Big|_0^{y=\frac{\pi}{3}}$

$= -\frac{1}{2} + 1 = \frac{1}{2}$

Choose the order of ~~iterated~~ integral

Recall:



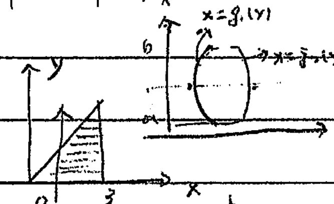
$\int_a^b \int_{g_2(x)}^{g_1(x)} f(x)\,dy\,dx$

Type II: Arbitrary regions

Keypoint: Graphing the region is helpful!

Example:

$\iint_R \frac{xy}{1+x^2+y^2}\,dA \qquad R = \{(x,y): 0 \le y \le x \quad 0 \le x \le 3\}$



$= \int_0^3 \int_0^x \frac{xy}{1+x^2+y^2}\,dy\,dx$ Let $u = 1+x^2+y^2$ $du = 2y\,dy$

substitution!

$\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dy$

$= \int_0^3 \frac{x}{2} \ln(1+x^2+y^2)\Big|_{y=0}^{y=x} dx$

$= \int_0^3 \frac{x}{2}\ln(1+2x^2) - \frac{x}{2}\ln(1+x^2)\,dx$ Let $V = 1+2x^2$ $du = 4x\,dx$ substitution!

integration by parts

$= \int \frac{1}{8} \ln u\,du - \int \frac{1}{4} \ln v\,dv$ $v = 1+x^2$ $dv = 2x\,dx$

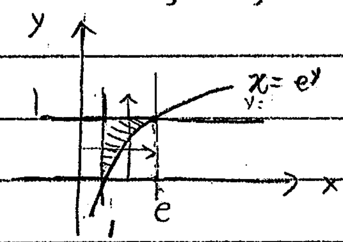
$= \frac{1}{8}(\ln u \cdot u - u) - [\frac{1}{4}(\ln v \cdot v - v)] = \frac{1}{8}(1+2x^2)\ln(1+2x^2) - \frac{1}{8}(1+2x^2) - \frac{1}{4}(1+x^2)\ln(1+x^2) + \frac{1}{4}(1+x^2)\Big|_0^3$

$= \frac{19}{8}\ln 19 - \frac{5}{2}\ln 10 - \frac{1}{8}$ #

Example: Page 1016 #60 change the order of integral

$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy$

$= \int_1^e \int_{\ln x}^1 f(x,y)\,dy\,dx$



$(\Leftrightarrow)\ y = \ln x$

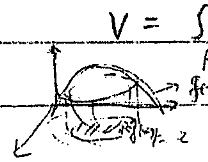
Topic 3: Applications of double integral

① Average value of a function over R. $\bar{f} = \frac{1}{\text{area of } R} \iint_R f\,dA$

② Calculate the volume between two surface:

$V = \iint_R (f(x,y) - g(x,y))\,dA$: $f(x,y)$: upper surface

$g(x,y)$: lower surface



R: projection of the intersections of f and g on xoy plane

③ Calculate area by double integral $A = \iint_R 1\,dA$ is the area of R.

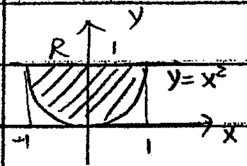
Example: #72 Page 1016.

Find the volume of the solid bounded by $z=x^2+1$ and $z=y+1$ and $y=1$

Answer: upper surface: $z=y+1$

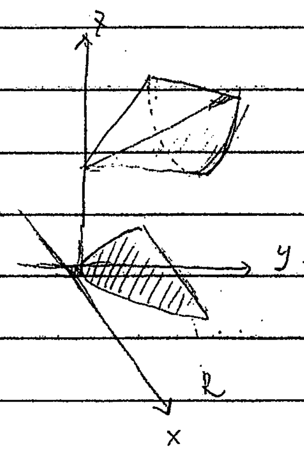
lower surface: $z=x^2+1$

intersection $\begin{cases} z=y+1 \\ z=x^2+1 \end{cases} \Rightarrow y+1=x^2+1 \Rightarrow y=x^2$ ~~but~~ $0\le y\le 1$ on xoy plane



$$V=\iint_R (y+1)-(x^2+1)\,dA.$$

$$=\int_{-1}^{1}\int_{x^2}^{1}(y+1)-(x^2+1)\,dy\,dx$$



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