Oct. 6

§13.6

Def. Let f be differentiable at (x,y). The gradient of f at (x,y) is

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ (A vector)

The directional derivative of f at a given point (a,b) in the direction of a given vector $\vec{u}$ is $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \frac{\vec{u}}{|\vec{u}|}$ (A number)

- Thm. 13.11 Let f be differentiable at (a,b) with $\nabla f(a,b) \neq 0$

steepest ascent ① f has max rate of increase at (a,b) in the direction of $\nabla f(a,b)$ and the rate of change is $|\nabla f(a,b)|$

steepest descent ② f ·· max ··· decrease ········· $-\nabla f(a,b)$ and ··· $-|\nabla f(a,b)|$

③ f has no change in any direction orthogonal to $\nabla f(a,b)$

- Thm 13.12 Let f be differentiable at (a,b) with $\nabla f(a,b) \neq 0$. The line tangent to the level curve at (a,b) is orthogonal to $\nabla f(a,b)$.

Example: §13.6 #53 (Similar to #10 in HW6)

$f(x,y) = 4 - x^2 - 2y^2$; $P(1,1,1)$

<1> Find the gradient of f and the gradient at P.

Answer: $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle -2x, -4y \rangle$

at P, $\nabla f(1,1) = \langle -2, -4 \rangle$

<2> Find the directional derivative at P in the direction of $\vec{u} = \langle -1, 2 \rangle$

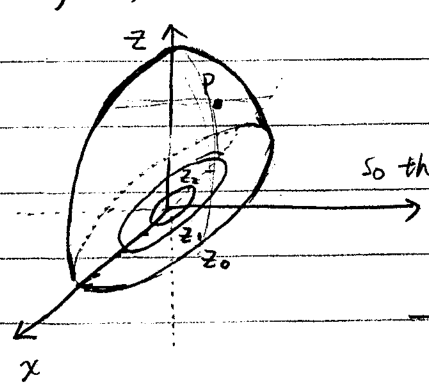
Answer: $D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \frac{\vec{u}}{|\vec{u}|}$

$= \langle -2, -4 \rangle \cdot \frac{\langle -1, 2 \rangle}{\sqrt{5}}$

$= \frac{-6}{\sqrt{5}}$ #

<3> What is the unit vector in the direction of steepest ascent at P?

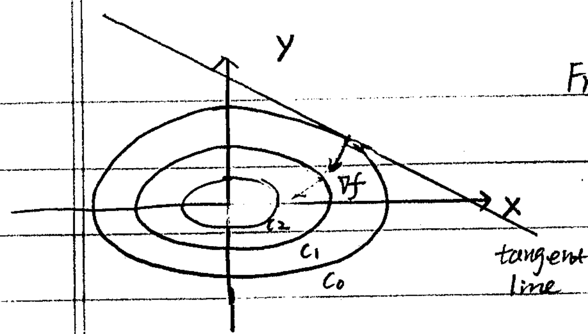
Can you find a vector in the direction of no change of the function at P?



Level curve through P:

level $= z_0 = f(1,1) = 4 - 1 - 2 = 1$

So the curve is $1 = 4 - x^2 - 2y^2 \Leftrightarrow \frac{x^2}{3} + \frac{y^2}{\frac{3}{2}} = 1$ (ellipse)



From level curve C_0 to C_2, f increases and ∇f provide the direction of steepest ascent. And ∇f is orthogonal to the tangent line at the point.

Answer: Unit vector in the direction of steepest ascent is $\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{\langle -2,-4\rangle}{\sqrt{20}}$

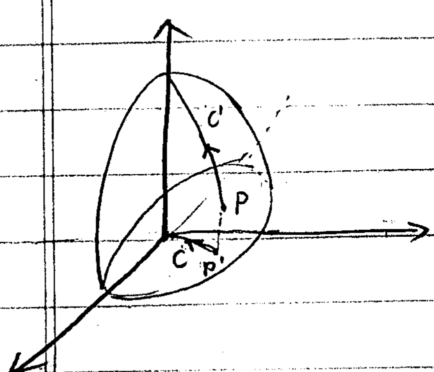
i.e. $-\frac{1}{\sqrt{5}}\langle 1,2\rangle$

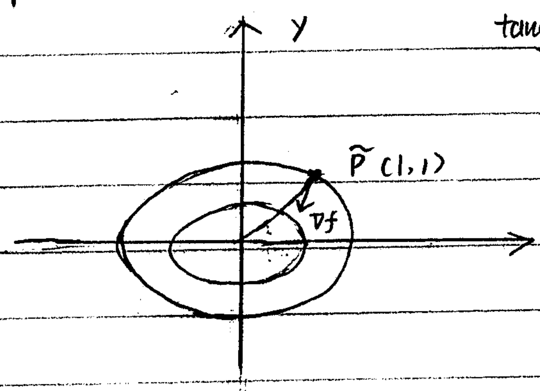
f has no change in any direction that is orthogonal to $\nabla f(1,1) = \langle -2,-4\rangle$

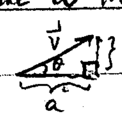
Simple examples are $\langle 4,-2\rangle$ and $\langle -4,2\rangle$ #

<4> Let c' be the path of steepest descent on the surface beginning at P and C is the projection of c' on xy-plane. Find the equation of C

Answer: The direction of steepest ascent is $\nabla f = \langle -2x, -4y\rangle$. So ∇f provides the tangent direction of C at any point (x,y).





It means at the point (x,y), the line tangent to the curve has slope $\frac{dy}{dx} = \frac{-4y}{-2x} = \frac{2y}{x}$ (Hint:  $\vec{v} = \langle a,b\rangle$ slope $= \frac{b}{a} = \tan\theta$)

$$\frac{dy}{y} = \frac{2dx}{x}$$

$$\ln y = 2\ln x + d = \ln x^2 + d$$

$$y = e^{\ln x^2 + d} = e^{\ln x^2}\cdot e^d = e^d x^2$$

Since $\tilde{P} = (1,1)$ is on the curve C, $e^d = 1$, so C has equation $y = x^2$.

<5> Find parametric equation for path c'

Answer: Let $x = t$, $y = t^2$ $z = f(x,y) = 4 - t^2 - 2(t^2)^2 = 4 - t^2 - 2t^4$

$\vec{r}(t) = \langle t, t^2, 4 - t^2 - 2t^4\rangle$ #