§13.3–13.5

1. Continuity:

Function f is continuous at point (a, b) if **all the following conditions are satisfied**:

① f is defined at (a, b) ② $\lim_{(x,y)\to(a,b)} f(x,y)$ exists ③ $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

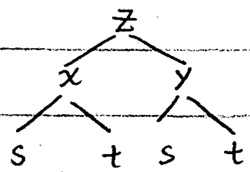
2. Partial derivatives. $f(x,y)$

f_x or $\frac{\partial f}{\partial x}$: treat y as a constant and take the usual derivative with respect to x.

f_y or $\frac{\partial f}{\partial y}$: treat x - - - - - - - - - - - - - - y.

3. Chain Rule: $z = f(x,y)$ and $x = x(s,t)$ $y = y(s,t)$.

Find partial derivative $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial s}$. $\boxed{\text{Use tree diagram}}$!



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

4. Implicit Differentiation:

Given $F(x,y) = 0$, the derivative $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Hint: Define y as a function of x $F(x, y(x)) = 0$, take derivative wrt x on both sides

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ and solve for $\frac{dy}{dx}$

5. Relation between continuity and differentiability.

Theorem: If f_x, f_y exist around (a,b), and are continuous at (a,b) then f is differentiable at (a,b).

Theorem: If f is differentiable at (a,b), then f is continuous at (a,b).

If f is not continuous at (a,b), then f is not differentiable at (a,b)

Examples:

1. §13.3 #39. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ At what points f are continuous?

Answer: ① f is continuous everywhere except $(0,0)$

② Now, check the continuity at point $(0,0)$

Condition 1: f is defined at $(0,0)$ ✓

Condition 2: check $\lim_{(x,y)\to(0,0)} f(x,y)$

Step 1: plug in ✗ Step 2. Simplify ✗

Step 3: Show the limit DNE

Pick path 1: $x=0$ $\lim_{y\to 0} \frac{0}{0+y^2} = 0$ > Different!

Pick path 2: $y=x$ $\lim_{y\to 0} \frac{y^2}{y^2+y^2} = \frac{1}{2}$.

By the two-path test, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE. Condition 2 is not satisfied

By ① and ②, $f(x,y)$ is continuous everywhere except $(0,0)$. #

2. #51. $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ Find the points where f is continuous

Answer: f is composite function $f(u) = \frac{\sin(u)}{u}$ and $u = x^2+y^2$

① $\lim_{u\to 0} f(u) = \lim_{u\to 0} \frac{\sin u}{u} = 1 = f(0)$, f is continuous at $u=0$

② $u = x^2+y^2$ is continuous at $(0,0)$

①+② $\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$

So $f(x,y)$ is continuous everywhere in $\mathbb{R}^2$. #

3. Find partial derivatives:

①. §13.4 #25

$G(s,t) = \frac{\sqrt{st}}{s+t}$

Answer: $\frac{\partial G}{\partial s} = \frac{\frac{1}{2}\sqrt{t}\frac{1}{\sqrt{s}}(s+t) - \sqrt{st}\cdot 1}{(s+t)^2} = \frac{\frac{\sqrt{t}}{2\sqrt{s}}(s+t) - \sqrt{st}}{(s+t)^2}$

$\frac{\partial G}{\partial t} = \frac{\frac{1}{2}\sqrt{s}\frac{1}{\sqrt{t}}(s+t) - \sqrt{st}\cdot 1}{(s+t)^2} = \frac{\frac{\sqrt{s}}{2\sqrt{t}}(s+t) - \sqrt{st}}{(s+t)^2}$

#35 $P(u,v) = \ln(u^2+v^2+4)$

Answer: $\frac{\partial P}{\partial u} = \frac{1}{u^2+v^2+4}\cdot 2u = \frac{2u}{u^2+v^2+4}$

$\frac{\partial P}{\partial v} = \frac{2v}{u^2+v^2+4}$

#54 $F(w,x,y,z) = w\sqrt{x^2+2y+3z}$

Answer: $\frac{\partial F}{\partial w} = \sqrt{x^2+2y+3z}$

$\frac{\partial F}{\partial x} = w\cdot\frac{1}{2}\cdot(x^2+2y+3z)^{-\frac{1}{2}}\cdot 2x = \frac{xw}{\sqrt{x^2+2y+3z}}$

$\frac{\partial F}{\partial y} = w\cdot\frac{1}{2}(x^2+2y+3z)^{-\frac{1}{2}}\cdot 2 = \frac{w}{\sqrt{x^2+2y+3z}}$

$\frac{\partial F}{\partial z} = w\,\frac{1}{2}(x^2+2y+3z)^{-\frac{1}{2}}\cdot 3 = \frac{3w}{2\sqrt{x^2+2y+3z}}$.

HW5 #17 $h(x,y,z) = (2+3x+y)^z$ Recall $(a^x)' = a^x\ln a$

$(x^a)' = ax^{a-1}$

Answer:

$$\frac{\partial h}{\partial x} = z(2+3x+y)^{z-1}\cdot 3 = 3z(2+3x+y)^{z-1}$$

$$\frac{\partial h}{\partial y} = z(2+3x+y)^{z-1}.$$

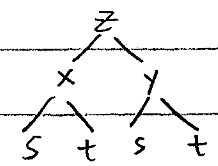
$$\frac{\partial h}{\partial z} = (2+3x+y)^{z}\cdot \ln(2+3x+y)$$

#

4 Chain Rule §13.5

#23. $z = e^{x+y}$ $x = st$ and $y = s+t$

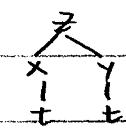
Answer:



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^{x+y}\cdot t + e^{x+y}\cdot 1 = e^{st+s+t}(t+1)$$

#HW5 #19 $z = \cos 3x \sin 3y$ and $x = \frac{t}{4}$ $y = t^5$

Answer



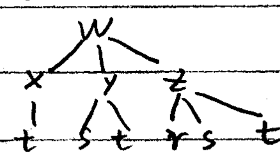
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= -3\sin 3x \sin 3y \cdot \frac{1}{4} + 3\cos 3x \cos 3y\,(5\cdot t^4)$$

$$= -\frac{3}{4}\sin\left(\frac{3t}{4}\right)\sin(3t^5) + 15\cos\left(\frac{3t}{4}\right)\cos(3t^5)$$ #

#28 $W = f(x,y,z)$ $x = g(t)$ $y = h(s,t)$ $z = p(r,s,t)$

Answer



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial w}{\partial x}\frac{dg}{dt} + \frac{\partial w}{\partial y}\frac{\partial h}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial p}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial h}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial p}{\partial s}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial w}{\partial z}\frac{\partial p}{\partial r}$$

5. Implicit Differentiation

#34. Find $\frac{dy}{dx}$, given $ye^{xy} - 2 = 0$

Answer: Let $F(x,y) = ye^{xy} - 2$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2e^{xy}}{e^{xy} + xye^{xy}} = -\frac{y^2}{1+xy}.$$

6. Differentiability. §13.4 # 57

$$f(x,y) = \begin{cases} -\frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

① Is f continuous at $(0,0)$?

Answer: Not continuous. Two-path test.

② Is f differentiable at $(0,0)$?

Answer: No, since f is not continuous at $(0,0)$

③ Evaluate $f_x(0,0)$, $f_y(0,0)$

Answer: $y=0$, $f(x,0)=0$, so $f_x(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = 0$

$x=0$, $f(0,y)=0$, so $f_y(0,0) = \frac{d}{dy} f(0,y)\Big|_{y=0} = 0$

④ Is f_x continuous at $(0,0)$?

For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{\partial}{\partial x}\left(-\frac{xy}{x^2+y^2}\right) = -\frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2} = -\frac{y^3 - x^2y}{(x^2+y^2)^2}$$

So

$$f_x(x,y) = \begin{cases} -\frac{y^3-x^2y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Take the path $y = 2x$,

$$\lim_{\substack{(x,y)\to(0,0) \\ y=2x}} f_x(x,y) = \lim_{x\to 0} -\frac{8x^3 - 2x^3}{(x^2+4x^2)^2} = -\lim_{x\to 0} \frac{6x^3}{25x^4} = -\lim_{x\to 0} \frac{6}{25x} \text{ DNE}$$

So $\lim_{(x,y)\to(0,0)} f_x(x,y)$ DNE and f_x is not continuous at $(0,0)$.