§ 12.8 – 12.9 Math 2153 Sep. 15

- Review

Assume $\vec{r}(t) = < x(t), y(t), z(t) >$ $a \le t \le b$ is a curve.

(1) Arc length function is the length of curve between $[a, t]$ for any t.

$$S(t) = \int_a^t |\vec{r}\,'(u)|\, du \qquad \left(\frac{ds}{dt} = |\vec{r}\,'(t)| \right)$$

(2) Arc length of the curve between $[a, b]$ is

$$L = \int_a^b |\vec{r}\,'(u)|\, du = S(b)$$

(3) If $|\vec{r}\,'(t)| = 1$, i.e. $\frac{ds}{dt} = 1$, then the parameter t corresponds to arc length s, and the curve is called parameterized by arc length. If $|\vec{r}\,'(t)| \ne 1$, then you can solve t in terms of s using $S(t)$ and reparameterize the curve in terms of arc length s

(4) Curvature: the rate of change of unit tangent $\vec{T}$ with respect to arc length, i.e. $\left|\frac{d\vec{T}}{ds}\right|$

Calculation: $k(t) = \frac{|d\vec{T}/dt|}{|\vec{r}\,'(t)|}$

$$k(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

(5) Principle Unit Normal vector: $\vec{N}(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

$\vec{N}$ is orthogonal to $\vec{T}$ and points to the inside of curve

(6) Decompose accerlation $\vec{a}$ into $\vec{N}$ and $\vec{T}$ directions

$$\vec{a} = a_N \vec{N} + a_T \vec{T} \quad \text{where} \quad a_N = k|\vec{r}\,'|^2 = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} \quad \text{and} \quad a_T = \frac{d|\vec{r}\,'(t)|}{dt} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{d^2 s}{dt^2}$$

(7) Unit Binormal vector: $\vec{B} = \vec{T} \times \vec{N}$

(8) Torsion: $\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N} = -\frac{d\vec{B}}{dt} \cdot \vec{N} \frac{1}{|\vec{r}\,'(t)|}$

The absolute value of τ is the rate that the curve twists out $\vec{N}$-$\vec{T}$ plane

(10) Arc length in polar coordinates

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

Example: $\vec{r}(t) = < 5\cos t^2, 5\sin t^2, 12t^2 >$ $\pi \le t \le 2\pi$

<1> Find arc length function $S(t)$

$$S(t) = \int_\pi^t |\vec{r}\,'(u)|\, du$$

$$\vec{r}\,'(t) = < -10t \sin t^2, 10t\cos t^2, 24t >$$

$$|\vec{r}'(t)| = \sqrt{(10t\sin t^2)^2 + (10t\cos t^2)^2 + (24t)^2}$$
$$= \sqrt{(10t)^2(\sin t^2)^2 + (10t)^2(\cos t^2)^2 + (24t)^2}$$
$$= \sqrt{(10t)^2[(\sin t^2)^2 + (\cos t^2)^2] + (24t)^2}$$
$$= \sqrt{100t^2 + 576t^2}$$
$$= 26t$$
$$S(t) = \int_\pi^t 26u\,du = 13u^2\Big|_\pi^t = 13(t^2 - \pi^2) \quad \#$$

2> Is the curve parameterized by arc length? If not, reparameterize it.

Answer: Since $|r'(t)| \neq 1$. Not parameterized by arc length

Set $S = 13(t^2 - \pi) \Rightarrow t^2 = \frac{S}{13} + \pi$

$$\vec{r}(t) = \vec{r}(s) = \langle 5\cos(\tfrac{S}{13}+\pi), 5\sin(\tfrac{S}{13}+\pi), 12(\tfrac{S}{13}+\pi) \rangle$$
$$= \langle -5\cos\tfrac{S}{13}, -5\sin\tfrac{S}{13}, 12(\tfrac{S}{13}+\pi) \rangle \quad \#$$

3> Find length of curve over $[\pi, 2\pi]$

Answer: $L = \int_\pi^{2\pi} |\vec{r}'(u)|\,du = S(2\pi) = 13((2\pi)^2 - \pi^2) = 39\pi^2$ #

4> Find curvature $K(t)$

Answer: $K = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -10t\sin t^2, 10t\cos t^2, 24t \rangle}{26t} = \langle -\tfrac{5}{13}\sin t^2, \tfrac{5}{13}\cos t^2, \tfrac{12}{13} \rangle$$
$$\frac{d\vec{T}}{dt} = \langle -\tfrac{10}{13}t\cos t^2, -\tfrac{10}{13}t\sin t^2, 0 \rangle$$
$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\tfrac{10}{13}t\cos t^2)^2 + (-\tfrac{10}{13}t\sin t^2)^2 + 0^2}$$
$$= \sqrt{(\tfrac{-10}{13}t)^2[(\cos t^2)^2 + (\sin t^2)^2]}$$
$$= \frac{10}{13}t$$
$$\text{So } K = \frac{\tfrac{10}{13}t}{26t} = \frac{5}{169} \quad \#$$

5> Find $\vec{N}(t)$

Answer: $\vec{N}(t) = \dfrac{d\vec{T}/dt}{|d\vec{T}/dt|} = \dfrac{\langle -\frac{10}{13}t\cos t^2, -\frac{10}{13}t\sin t^2, 0 \rangle}{\frac{10}{13}t}$

$$= \langle -\cos t^2, -\sin t^2, 0 \rangle$$

6> Decompose $\vec{a}$ in $\vec{N}$ and $\vec{T}$ directions

Answer: $a_N = K \cdot |r'(t)|^2 = \frac{5}{169} \cdot (26t)^2 = 20t^2$

$a_T = \frac{d}{dt}|r'(t)| = \frac{d}{dt}(26t) = 26$

$\vec{a} = 20t^2\vec{N} + 26\vec{T}$ #

<7> Find binomal vector $\vec{B}$ and torsion τ

Answer: $\vec{B} = \vec{T} \times \vec{N}$

$$= \begin{vmatrix} i & j & k \\ -\frac{5}{13}\sin t^2 & \frac{5}{13}\cos t^2 & \frac{12}{13} \\ -\cos t^2 & -\sin t^2 & 0 \end{vmatrix}$$

$$= \frac{12}{13}\sin t^2\,\vec{i} - \frac{12}{13}\cos t^2\,\vec{j} + \frac{5}{13}\vec{k}$$

$$= \langle \frac{12}{13}\sin t^2, -\frac{12}{13}\cos t^2, \frac{5}{13} \rangle$$

$\tau = -\frac{dB}{dt}\cdot\vec{N}\,\frac{1}{|\vec{r}'(t)|}$

$\frac{dB}{dt} = \langle \frac{24}{13}t\cos t^2, \frac{24}{13}t\sin t^2, 0 \rangle$

$\frac{dB}{dt}\cdot N = -\frac{24}{13}t(\cos t^2)^2 - \frac{24}{13}t(\sin t^2)^2 = -\frac{24}{13}t$

$\tau = -(-\frac{24}{13}t)\cdot\frac{1}{26t} = \frac{12}{169}$ #