Sep. 8

§ 12.5 & 12.6

• Vector-valued function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(t)$ can be viewed as a 3D curve, representing the path of a moving particle.

Classic Problems.

<1> Tangent vector: General Form: $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

At specific point: $\vec{r}'(t_0)$ plug in t_0

Unit tangent vector: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

<2> Intersecting curves and colliding particles.

Book §12.5 #68 $\vec{r}(t) = \langle 2+2t, 8+t, 10+3t \rangle$

$\vec{R}(s) = \langle 6+s, 10-2s, 16-s \rangle$

(a) Determine whether the lines intersect. (common point) and find the point

(b). If. $\vec{r}(t)$ and $\vec{R}(s)$ are paths of two particles, do they collide?

Answer:

(a) Intersection $\Leftrightarrow$ common point.

$$\begin{cases} 2+2t = 6+s & ① \\ 8+t = 10-2s & ② \\ 10+3t = 16-s & ③ \end{cases}$$

①&② $\Rightarrow \begin{cases} t = 2 \\ s = 0 \end{cases}$

plug in ③ to check $t=2, s=0$ is a solution

Coordinates of intersection: $\vec{r}(2) = \langle 6, 10, 16 \rangle = \vec{R}(0)$

(b). Collide $\Leftrightarrow$ Two particles reach the same point at the same time

From (a), we know that there is only one common point on two paths.

The first particle reaches it at $t=2$, but the second particle gets it at $s=0$. So they do not collide. #

<3> $\lim_{t \to t_0} \vec{u}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle$

$\frac{d}{dt} \vec{u}(t) = \langle x'(t), y'(t), z'(t) \rangle$

$\frac{d}{dt} \vec{u}(f(t)) = \vec{u}'(f(t)) \cdot f'(t)$

product rules

$$\begin{cases} \frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \\ \frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t) \end{cases}$$

product rules | $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

✩ $\int \vec{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle + \langle C_1, C_2, C_3 \rangle$

Examples:

$\vec{r}'(t) = \langle te^{-t}, -\cos t, \sec^2 t \rangle \qquad \vec{r}(0) = \langle 5, 5, 5 \rangle$

Question: Find $\vec{r}(t)$.

Answer: $\vec{r}(t) = \int \vec{r}'(t)\,dt$

$= \langle \int te^{-t}\,dt, \int -\cos t\,dt, \int \sec^2 t\,dt \rangle + \boxed{\langle C_1, C_2, C_3 \rangle}$

Use $\vec{r}(0)$ to **SOLVE** for C_1, C_2, C_3

$\int te^{-t}\,dt$: Recall: Integration by parts $\int u\,dv = uv - \int v\,du$

Let $u = t \quad dv = e^{-t}dt$ The order to choose u: LIATE

$du = dt \quad v = -e^{-t}$

$\int te^{-t}\,dt = t\cdot(-e^{-t}) - \int -e^{-t}\,dt = -te^{-t} - e^{-t} + C_1$

$\int -\cos t\,dt = -\sin t + C_2$

$\int \sec^2 dt = \tan t + C_3$

$\vec{r}(t) = \langle -te^{-t} - e^{-t}, -\sin t, \tan t \rangle + \langle C_1, C_2, C_3 \rangle$

Use $\vec{r}(0) = \langle 5, 5, 5 \rangle$ to solve for C_1, C_2, C_3

$\vec{r}(0) = \langle 0 - e^0, -\sin 0, \tan 0 \rangle + \langle C_1, C_2, C_3 \rangle = \langle 5, 5, 5 \rangle$

$\langle -1, 0, 0 \rangle + \langle C_1, C_2, C_3 \rangle = \langle 5, 5, 5 \rangle$

$C_1 = 6 \quad C_2 = 5 \quad C_3 = 5$ #

§12.7 Motion in Space:

Position $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity $\vec{v}(t) = \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Acceleration $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

(acceleration → position: initial $\vec{r}(0)$ + initial $\vec{v}(0)$)

Example: $\vec{a}(t) = \langle \cos t, 5\sin t \rangle$ initial position $\langle x_0, y_0 \rangle = \langle 1, 0 \rangle$, initial velocity $\langle 0, 5 \rangle$

Find $\vec{v}(t), \vec{r}(t)$

Answer: $\vec{v}(t) = \int a(t)\,dt + \vec{C} = \langle \int \cos t\,dt, \int 5\sin t\,dt \rangle + \langle C_1, C_2 \rangle$

$= \langle \sin t, -5\cos t \rangle + \langle C_1, C_2 \rangle$

$\vec{v}(0) = \langle \sin 0, -5\cos 0 \rangle + \langle C_1, C_2 \rangle = \langle 0, 5 \rangle \Rightarrow C_1 = 0, C_2 = 10$

$\vec{v}(t) = \langle \sin t, -5\cos t + 10 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{c}$

$= \langle \int \sin t\,dt, \int -5\cos t + 10\,dt \rangle + \langle d_1, d_2 \rangle$

$= \langle -\cos t, -5\sin t + 10t \rangle + \langle d_1, d_2 \rangle$

$\vec{r}(0) = \langle -\cos 0, -5\sin 0 + 0 \rangle + \langle d_1, d_2 \rangle = \langle 1, 0 \rangle \Rightarrow d_1 = 2, d_2 = 0$

$\vec{r}(t) = \langle -\cos t + 2, -5\sin t + 10t \rangle$ #

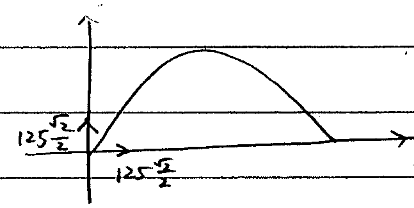
HW3 #6 Initial position $\langle 0, 0 \rangle$, initial speed $|\vec{v}(0)| = 125$ m/s. launch angle $\alpha = 45°$

Assuming no forces other than gravity

Q: Find object in the air for ____ seconds?

Find the range of object. g m/s² ↓

Find the maximum height



Answer: $\vec{a}(t) = \langle 0, -g \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 0, -gt \rangle + \langle c_1, c_2 \rangle$

$\vec{v}(0) = \langle c_1, c_2 \rangle = \langle 125\frac{\sqrt{2}}{2}, 125\frac{\sqrt{2}}{2} \rangle$

- $\vec{v}(t) = \langle 125\frac{\sqrt{2}}{2}, -gt + 125\frac{\sqrt{2}}{2} \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 125\frac{\sqrt{2}}{2}t, -\frac{1}{2}gt^2 + 125\frac{\sqrt{2}}{2}t \rangle + \langle c_1, c_2 \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle + \langle c_1, c_2 \rangle = \langle 0, 0 \rangle$

- $\vec{r}(t) = \langle 125\frac{\sqrt{2}}{2}t, -\frac{1}{2}gt^2 + 125\frac{\sqrt{2}}{2}t \rangle = \langle x(t), y(t) \rangle$

① Hit the ground $y(t) = 0 \Leftrightarrow -\frac{1}{2}gt^2 + 125\frac{\sqrt{2}}{2}t = 0$ $t = 0$ or $t = \frac{125\sqrt{2}}{g}$

The time in the air is $\frac{125\sqrt{2}}{g} = T_1$

② Range = the horizontal position when the object hits ground

$= x(T_1)$, where $T_1 = \frac{125\sqrt{2}}{g}$

③ Maximum height: when vertical velocity $= 0 \Leftrightarrow -gt + 125\frac{\sqrt{2}}{2} = 0$

$T_2 = \frac{125\sqrt{2}}{2g}$

The corresponding height is $y(T_2)$ #