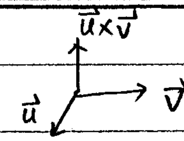
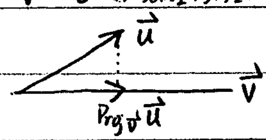
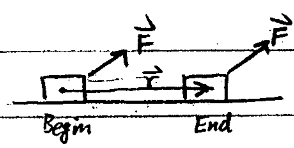
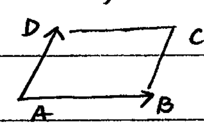
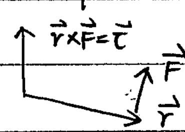
Sep 1 | Math 2153 | HW1 due on Sunday, Sep. 4

§ 12.2 — 12.4

- Summary of dot product and cross product

Given $\vec{u} = \langle x_1, y_1, z_1 \rangle$ $\quad \vec{v} = \langle x_2, y_2, z_2 \rangle$

	Dot Product	Cross Product
Def:	$\vec{u}\cdot\vec{v} = \lvert\vec{u}\rvert\cdot\lvert\vec{v}\rvert\cdot\cos\theta$ $\vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 + z_1\cdot z_2$ $\vec{u}\cdot\vec{v}$ is a scalar, a number	① $\lvert\vec{u}\times\vec{v}\rvert = \lvert\vec{u}\rvert\lvert\vec{v}\rvert\cdot\sin\theta$  ② Direction: right hand rule $\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ $= (y_1z_2 - y_2z_1)\vec{i} - (x_1z_2 - x_2z_1)\vec{j} + (x_1y_2 - x_2y_1)\vec{k}$ $\vec{u}\times\vec{v}$ is a vector !
Related Problems	① Parallel and orthogonal · $\vec{u} \parallel \vec{v} \Leftrightarrow \vec{u} = \lambda\vec{v} \Leftrightarrow \begin{cases} x_1 = \lambda x_2 \\ y_1 = \lambda y_2 \\ z_1 = \lambda z_2 \end{cases}$ · $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u}\cdot\vec{v} = 0 \Leftrightarrow x_1x_2 + y_1y_2 + z_1z_2 = 0$ ② Projection  $Prj_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\lvert\vec{v}\rvert}\cdot\frac{\vec{v}}{\lvert\vec{v}\rvert}$, where $Scal_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\lvert\vec{v}\rvert}$ and $\frac{\vec{v}}{\lvert\vec{v}\rvert}$ is the unit vector in $\vec{v}$ direction ③ Applications in physics Work done by force $W = \vec{F}\cdot\vec{r}$ 	① Area of parallelogram  Area $ABCD = \lvert\vec{AB}\times\vec{AD}\rvert$ ② Applications in physics a. Torque (twisting effect)  $\vec{\tau} = \vec{r}\times\vec{F}$ b. Magnetic force on moving change. $\vec{F} = q\cdot(\vec{v}\times\vec{B})$

§ 12.5

Equation of line passing through $P_0(x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle a, b, c\rangle$ is

(vector equation) $\quad \langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \qquad -\infty < t < +\infty$

Equivalently, the parametric equation is

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct \qquad -\infty < t < +\infty$$

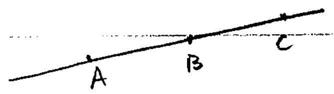
Keypoints: ① One point on the line ② the direction !

Practice:

§ 12.2 – 12.4

✓ HW1 #11 Determine x, y such that $\overset{A}{(1,2,3)}$ $\overset{B}{(8,1,1)}$ and $\overset{C}{(x,y,2)}$ are collinear.

Answer:



A, B, C are collinear $\iff \overrightarrow{AB} \parallel \overrightarrow{BC}$

$\overrightarrow{AB} = \langle 7, -1, -2 \rangle \qquad \overrightarrow{BC} = \langle x-8, y-1, 1 \rangle$

$\overrightarrow{AB} \parallel \overrightarrow{BC} \iff$ there exists λ such that $\overrightarrow{BC} = \lambda \overrightarrow{AB} \iff$

$$\begin{cases} x-8 = \lambda \cdot 7 & ① \\ y-1 = \lambda \cdot (-1) & ② \\ 1 = \lambda \cdot (-2) & ③ \Rightarrow \lambda = -\frac{1}{2} \end{cases}$$

plug in

$$\Rightarrow \begin{cases} x = \frac{9}{2} \\ y = \frac{3}{2} \end{cases}$$

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HW1 #13 $\vec{v} = -3\vec{i} + \vec{j} + 4\vec{k} \qquad \vec{w} = \vec{i} + 2\vec{j} + 3\vec{k}$. Find angle between them.

Answer: Recall $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cdot \cos\theta$

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right)$$

$\vec{v} \cdot \vec{w} = \langle -3, 1, 4 \rangle \cdot \langle 1, 2, 3 \rangle = -3 + 2 + 12 = 11$

$|\vec{v}| = \sqrt{9+1+16} = \sqrt{26} \qquad |\vec{w}| = \sqrt{1+4+9} = \sqrt{14}$

$\theta = \cos^{-1} \frac{11}{\sqrt{364}}$

HW1 #15. $\vec{u} = 2\vec{i} + 2\vec{j} + 2\vec{k} \qquad \vec{v} = 4\vec{i} - 3\vec{j} + \vec{k}$. Calculate. $\text{Proj}_{\vec{v}}\vec{u}$, $\text{Scal}_{\vec{v}}\vec{u}$

Answer: Recall. $\text{Proj}_{\vec{v}}\vec{u} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}_{\text{Scal}_{\vec{v}}\vec{u}} \cdot \frac{\vec{v}}{|\vec{v}|}$

$\vec{u} \cdot \vec{v} = \langle 2, 2, 2 \rangle \cdot \langle 4, -3, 1 \rangle = 8 - 6 + 2 = 4$

$|\vec{v}| = \sqrt{16+9+1} = \sqrt{26}$

$\text{Scal}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{4}{\sqrt{26}}$

$\text{Proj}_{\vec{v}}\vec{u} = \frac{4}{\sqrt{26}} \frac{\langle 4, -3, 1 \rangle}{\sqrt{26}} = \frac{2}{13} \langle 4, -3, 1 \rangle$

$= \langle \frac{8}{13}, \frac{-6}{13}, \frac{2}{13} \rangle$

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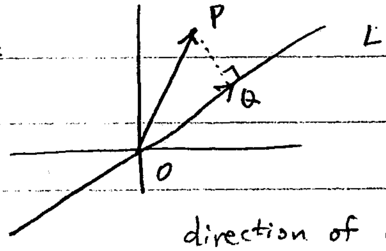
✓ HW1 #18. Find the distance between a point P and a line L.

The line L passes the origin and has direction $\langle -4, -9, -4 \rangle$.

The point is $P(-1, 1, -1)$

Answer:



The distance between P and L is $|\overrightarrow{QP}|$

$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} \Rightarrow \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$\overrightarrow{OP} = \langle -1, 1, -1 \rangle \qquad \overrightarrow{OQ} = \mathrm{Proj}_L \overrightarrow{OP}$

direction of Line L $\vec{v} = \langle -4, -9, -4 \rangle$

$$\mathrm{Proj}_L \overrightarrow{OP} = \frac{\overrightarrow{OP}\cdot\vec{v}}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|} = \frac{\overrightarrow{OP}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}$$

$$\overrightarrow{OP}\cdot\vec{v} = \langle -1,1,-1\rangle \cdot \langle -4,-9,-4\rangle = -1$$

$$|\vec{v}|^2 = 16+81+16 = 113$$

$$\overrightarrow{OQ} = \mathrm{Proj}_L \overrightarrow{OP} = \frac{-1}{113}\times\vec{v} = \frac{-1}{113}\langle -4,-9,-4\rangle = \left\langle \frac{4}{113}, \frac{9}{113}, \frac{4}{113}\right\rangle$$

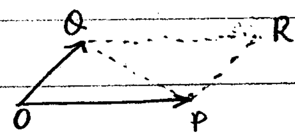
$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \langle -1,1,-1\rangle - \left\langle \frac{4}{113}, \frac{9}{113}, \frac{4}{113}\right\rangle$$

$$= \left\langle \frac{-117}{113}, \frac{104}{113}, \frac{-117}{113}\right\rangle$$

$|\overrightarrow{QP}| = \sqrt{\cdots\cdots}$ #

HW2 # 4. Find the area of triangle with vertices $O(0,0,0)$, $P(1,1,4)$, $Q(6,5,3)$

Answer:



area$(\triangle OPQ) = \frac{1}{2}$ area$(\square_{OPRQ}) = \frac{1}{2}|\overrightarrow{OP}\times\overrightarrow{OQ}|$

$\overrightarrow{OP} = \langle 1, 1, 4 \rangle \qquad \overrightarrow{OQ} = \langle 6, 5, 3 \rangle$

$$\overrightarrow{OP}\times\overrightarrow{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ 6 & 5 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 4 \\ 6 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix}\vec{k}$$

$$= (3-20)\vec{i} - (3-24)\vec{j} + (5-6)\vec{k}$$

$$= -17\vec{i} + 21\vec{j} - \vec{k}$$

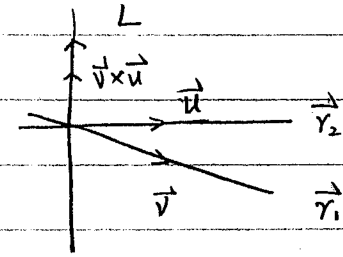
$$|\overrightarrow{OP}\times\overrightarrow{OQ}| = \sqrt{17^2+21^2+1} = \sqrt{731}$$

area$(\triangle OPQ) = \frac{1}{2}\sqrt{731}$ #

HW2 #9. Find equation of Line through $(1,2,6)$ that is normal to both

$\vec{r_1}(t) = \langle 3-3t, 1+9t, 6-6t \rangle$ and $\vec{r_2}(t) = \langle -3t, 1+t, 6-t \rangle$.

Answer: Key point: Find the direction!



The direction of $\vec{r_1}$ is $\langle -3, 9, -6 \rangle = \vec{v}$

The direction of $\vec{r_2}$ is $\langle -3, 1, -1 \rangle = \vec{u}$

The direction of L is $\vec{v} \times \vec{u}$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 9 & -6 \\ -3 & 1 & -1 \end{vmatrix} = (9\cdot(-1) - (1)\cdot(-6))\vec{i} - [(-3)(-1) - (-3)(-6)]\vec{j} + ((-3)(1) - (-3)\cdot 9)\vec{k}$$

$$= -3\vec{i} + 15\vec{j} + 24\vec{k}$$

The direction is $\langle -3, 15, 24 \rangle$.

The equation of L is

$$\vec{r}(t) = \langle 1, 2, 6 \rangle + \langle -3, 15, 24 \rangle \cdot t$$

$\Leftrightarrow$

$$\begin{cases} x(t) = 1 - 3t \\ y(t) = 2 + 15t \\ z(t) = 6 + 24t \end{cases}$$

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