Math 2153 - Autumn 2016 Name (Please print): ______________

Quiz 9 Recitation time (Please circle): 1:50 3:00 4:10

SHOW ALL WORK FOR THE PROBLEMS!!! Unsupported answers might not receive full credit.

Problem 1 [5 pts] Evaluate the following integral **by using spherical coordinates**

$$\iiint_D e^{-(x^2+y^2+z^2)^{3/2}} dV,$$

where D is the unit ball.

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho\cos\varphi \end{cases}, \quad dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta, \quad x^2+y^2+z^2 = \rho^2$$

$$D = \{ (x,y,z) \mid x^2+y^2+z^2 \le 1 \}$$
$$= \{ (\rho, \varphi, \theta) \mid 0 \le \rho \le 1, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi \}$$

So the original integral becomes

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-(\rho^2)^{3/2}} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-\rho^3} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\underset{du = 3\rho^2 d\rho}{\overset{u=\rho^3}{=}} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \frac{1}{3} e^{-u} \sin\varphi \, du \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(-\frac{1}{3} \sin\varphi \cdot e^{-u} \right) \Big|_{u=0}^{u=1} d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{1}{3}(1-e^{-1}) \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}(1-e^{-1}) \int_0^{2\pi} (-\cos\varphi) \Big|_{\varphi=0}^{\varphi=\pi} d\theta$$

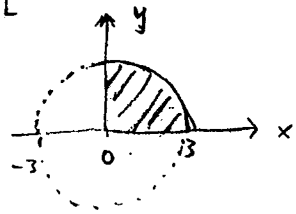
$$= \frac{1}{3}(1-e^{-1}) \int_0^{2\pi} 2 \, d\theta = \frac{2}{3}(1-e^{-1}) \cdot 2\pi = \frac{4\pi}{3}(1-e^{-1})$$

(There is another problem on the back.)

Problem 2 [5 pts] Evaluate the following integral by using cylindrical coordinates

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} (x^2+y^2)^{-1/2} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$$

The region in question is bounded from above by $z = \sqrt{x^2+y^2}$ and from below by $z=0$. The projection of the region onto the xy-plane is $\{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2}\}$ which is the quarter disk (shaded part)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$dy\,dx = r\,dr\,d\theta$, $x^2+y^2 = r^2$

the half disk $= \{(r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2}\}$

So the original integral becomes

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^r (r^2)^{-\frac{1}{2}}\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^r r^{-1} \cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^r 1\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} r^2 \Big|_{r=0}^{r=3}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{9}{2}\, d\theta$$

$$= \frac{9}{4}\pi$$