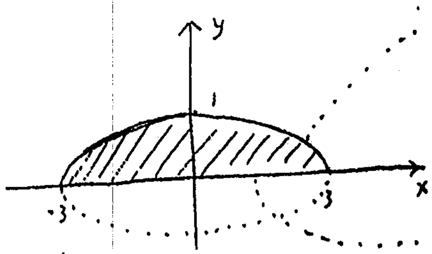
Math 2153 - Autumn 2016 Name (Please print): ______________

Quiz 10 Recitation time (Please circle): 1:50 3:00 4:10

SHOW ALL WORK FOR THE PROBLEMS!!! Unsupported answers might not receive full credit.

Problem 1 [4 pts] Find the center of mass $(\bar{x}, \bar{y})$ of the upper half ($y \geq 0$) of the plate bounded by the ellipse $x^2 + 9y^2 = 9$ with density $\rho(x, y) = 1 + y$. (Hint: since the region is symmetric with respect to the y-axis and the density ρ does not rely on x, $\bar{x} = 0$. Then you only need to find $\bar{y}$.) Do Not Evaluate the involved integrals but you need to write the integrals as iterated double integrals.

Let us denote the region in question by R (shown in the figure below).



upper curve $y = \sqrt{1 - \frac{x^2}{9}}$

(solved from $x^2 + 9y^2 = 9$)

lower curve $y = 0$

Then

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

$$= \frac{\int_{-3}^{3} \int_0^{\sqrt{1-\frac{x^2}{9}}} y(1+y)\,dy\,dx}{\int_{-3}^{3} \int_0^{\sqrt{1-\frac{x^2}{9}}} (1+y)\,dy\,dx}$$

or $$= \frac{\int_0^1 \int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} y(1+y)\,dx\,dy}{\int_0^1 \int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} (1+y)\,dx\,dy}$$

(There is another problem on the back.)

<u>Problem 2</u> [6 pts] Evaluate the following integrals using a change of variables

$$\iint_R \frac{y-x}{y+2x+1}\,\mathrm{d}A,$$

where R is the parallelogram bounded by $y-x=1$, $y-x=2$, $y+2x=0$, and $y+2x=4$.

Let $\begin{cases} u = y-x & \cdots ① \\ v = y+2x & \cdots ② \end{cases}$. Then $1 \le u \le 2$ and $0 \le v \le 4$

and the new region is $R' = \{(u,v) \mid 1 \le u \le 2,\ 0 \le v \le 4\}$.

To compute $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$, we need to solve ① & ② for

x and y. ② − ① gives $3x = v-u$ and $x = -\frac{1}{3}u + \frac{1}{3}v$.

Plug $x = -\frac{1}{3}u + \frac{1}{3}v$ in ① to obtain $y = u + x = \frac{2}{3}u + \frac{1}{3}v$. So

$\begin{cases} x = -\frac{1}{3}u + \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$ and $\frac{\partial x}{\partial u} = -\frac{1}{3}$, $\frac{\partial y}{\partial u} = \frac{2}{3}$, $\frac{\partial x}{\partial v} = \frac{1}{3}$, $\frac{\partial y}{\partial v} = \frac{1}{3}$

and

$$J(u,v) = \begin{vmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}\cdot\frac{1}{3} - \frac{2}{3}\cdot\frac{1}{3} = -\frac{1}{3}.$$

Then

$$\iint_R \frac{y-x}{y+2x+1}\,dA = \iint_{R'} \frac{u}{v+1}\,|J(u,v)|\,dA$$

$$= \int_1^2 \int_0^4 \frac{u}{v+1}\,\frac{1}{3}\,dv\,du = \int_1^2 \frac{1}{3}u \ln|v+1| \Big|_0^{v=4} du$$

$$= \int_1^2 \frac{\ln 5}{3}\,u\,du = \frac{\ln 5}{3}\cdot\frac{1}{2}u^2\Big|_1^2 = \frac{\ln 5}{3}\cdot\frac{3}{2} = \frac{\ln 5}{2}$$