Math 2153 **Name:**____________________________

Midterm 2 **Recitation Instructor:**________________

November 18, 2016 **Recitation Time:** ____________________

The point value of each problem is indicated. To obtain full credit, you must have the correct answers along with the supporting work. Incorrect answers with work shown may receive partial credit, but **answers without supporting work will receive NO credit**, except for choice problems.

The exam consists of 5 problems starting on page 2 and ending on page 7. Make sure your exam is not missing any pages before you start.

Scientific calculators are allowed. Graphing calculators are NOT permitted.

Pros	Score
1	(25)
2	(14)
3	(25)
4	(16)
5	(20)
Total	(100)

D

1. (25 pts) Multiple Choices: **Circle your answer.**

(i) (5 pts) Which integral can be used to find the area of the region $R = \{(x, y) : c \leq y \leq d,\ g(y) \leq x \leq h(y)\}$ in xy-plane?

(a) $\displaystyle\int_{g(y)}^{h(y)} \int_c^d \, dx\, dy$ (b) $\displaystyle\int_{g(y)}^{h(y)} \int_c^d \, dy\, dx$

(c) $\displaystyle\int_c^d \int_{g(y)}^{h(y)} \, dy\, dx$ (d) $\displaystyle\int_c^d \int_{g(y)}^{h(y)} \, dx\, dy$

D

(ii) (5 pts) If we change the order of integration, $\displaystyle\int_0^2 \int_{x^2}^{2x} f(x, y)\, dy\, dx$ is equal to

(a) $\displaystyle\int_{x^2}^{2x} \int_0^2 f(x, y)\, dx\, dy$ (b) $\displaystyle\int_0^2 \int_{2y}^{y^2} f(x, y)\, dx\, dy$

(c) $\displaystyle\int_0^4 \int_{\sqrt{y}}^{2y} f(x, y)\, dx\, dy$ (d) $\displaystyle\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x, y)\, dx\, dy$

B

(iii) (5 pts) To find the volume of the solid $D = \{(x, y, z) : g(x, y) \leq z \leq f(x, y),\ (x, y) \in R\}$ where R is a region in xy-plane, which of the following formula is **wrong**?

(a) $\displaystyle\iiint_D 1\, dV$

(b) $\displaystyle\iiint_D (f(x, y) - g(x, y))\, dV$

(c) $\displaystyle\iint_R \int_{g(x,y)}^{f(x,y)} 1\, dz\, dA$

(d) $\displaystyle\iint_R \{f(x, y) - g(x, y)\}\, dA$

C (iv) (5 pts) Given that $P = (0, 1)$ is a critical point of $f(x, y) = 2y\cos x - y^2$, it is a

(a) Saddle point (b) Local minimum

(c) Local maximum (d) Inconclusive case

A (v) (5 pts) If $u = xy$, $v = y$, the Jacobian $J(u, v) = \dfrac{\partial(x, y)}{\partial(u, v)} =$

(a) $\dfrac{1}{v}$ (b) 1 (c) $-\dfrac{u}{v^2}$ (d) 0

2. (14 pts) Consider the function $f(x, y) = \ln(x^2 + y^2 + 1)$ and $p = (1, 1)$.

(a) (8 pts) Compute the directional derivative of f at p in the direction of $\langle 3, -4\rangle$.

(b) (6 pts) Find the unit vectors that give the direction of steepest ascent and steepest descent at p.

a). $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \frac{\vec{u}}{|\vec{u}|}$ Here $(a,b) = (1,1)$

$\vec{u} = \langle 3, -4 \rangle$

$f_x = \frac{2x}{x^2+y^2+1}$, $f_y = \frac{2y}{x^2+y^2+1}$ $\frac{\vec{u}}{|\vec{u}|} = \frac{\langle 3, -4\rangle}{\sqrt{3^2+4^2}} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

So $\nabla f(1,1) = \langle \frac{2\cdot 1}{1^2+1^2+1}, \frac{2\cdot 1}{1^2+1^2+1} \rangle = \langle \frac{2}{3}, \frac{2}{3} \rangle$

$D_{\vec{u}} f(1,1) = \langle \frac{2}{3}, \frac{2}{3} \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{2}{3}\cdot\frac{3}{5} + \frac{2}{3}\cdot\frac{-4}{5} = \frac{6-8}{15} = -\frac{2}{15}$

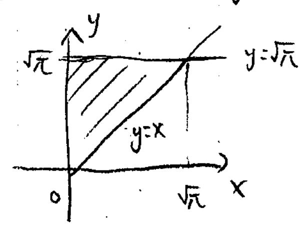
(b) Steepest ascent has direction $\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{\langle \frac{2}{3}, \frac{2}{3}\rangle}{\sqrt{(\frac{2}{3})^2+(\frac{2}{3})^2}} = \frac{\langle \frac{2}{3}, \frac{2}{3}\rangle}{\frac{2}{3}\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Steepest descent has direction $-\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

3. (25 pts) Evaluate the integrals.

(i) (13 pts) $\iint\limits_R \sin(y^2)\, dA$, where R is the region bounded by $y = x$, $y = \sqrt{\pi}$ and the positive y axis.

R is the /// region



$$\iint\limits_R \sin(y^2)\, dA = \int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} \sin(y^2)\, dy\, dx \quad \left(\begin{array}{c}\text{but difficult}\\ \text{to integrate}\end{array}\right)$$

$$= \int_0^{\sqrt{\pi}} \int_0^{y} \sin(y^2)\, dx\, dy = \int_0^{\sqrt{\pi}} \left(x\sin(y^2) \Big|_{x=0}^{x=y} \right) dy$$

$$= \int_0^{\sqrt{\pi}} y\sin(y^2)\, dy$$

$$\left(\begin{array}{l} \int y\sin(y^2)\, dy \\ = \frac{1}{2}\int \sin u\, du \\ = -\frac{1}{2}\cos u = -\frac{1}{2}\cos y^2 \end{array} \right) \leftarrow \left(\begin{array}{l} u = y^2 \\ du = 2y\, dy \end{array} \right)$$

$$= -\frac{1}{2}\cos(y^2)\Big|_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\cos\pi - \left(-\frac{1}{2}\cos 0\right)$$

$$= -\frac{1}{2}\cdot(-1) - \left(-\frac{1}{2}\cdot 1\right) = \frac{1}{2} + \frac{1}{2} = 1$$

(ii) (12 pts) $\iiint\limits_D \sqrt{x^2+y^2+z^2}\, dV$, where D is the **upper half ball** with radius 2.

In spherical coordinates, $D = \{(\rho, \varphi, \theta) : 0 \le \rho \le 2,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}$

with $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$, we have $x^2+y^2+z^2 = \rho^2$

So the integral in question is

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{\rho^2}\ \overbrace{\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}^{dV}$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^3 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi\right) \left(\int_0^2 \rho^3\, d\rho\right)$$

$$= 2\pi \left(-\cos\varphi\Big|_0^{\frac{\pi}{2}}\right) \left(\frac{1}{4}\rho^3\Big|_0^2\right)$$

$$= 2\pi \cdot 1 \cdot \frac{16}{4} = 8\pi$$

4. (16 pts) Use Lagrange multiplier to find the maximum and minimum of $f(x, y) = x + 4y$ with (x, y) satisfying $x^2 + y^2 - xy = 7$. Set $g(x,y) = x^2+y^2-xy-7$

$$\text{Set} \begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}, \qquad f_x = 1, \; f_y = 4, \qquad g_x = 2x-y, \; g_y = 2y-x$$

$$\Leftrightarrow \begin{cases} 1 = \lambda \cdot (2x-y) \cdots ① \\ 4 = \lambda(2y-x) \cdots ② \\ x^2+y^2-xy = 7 \cdots ③ \end{cases} \qquad (①, ② \Rightarrow \lambda \neq 0, \; 2x-y \neq 0, \; 2y-x \neq 0.)$$

Approach (i): $\frac{①}{②} \Rightarrow \frac{1}{4} = \frac{2x-y}{2y-x} \Leftrightarrow 2y-x = 8x-4y \Leftrightarrow 6y = 9x$
$\Leftrightarrow y = \frac{3}{2}x$

Plug $y = \frac{3}{2}x$ in ③ $\Rightarrow x^2 + \frac{9}{4}x^2 - \frac{3}{2}x^2 = 7 \Leftrightarrow \frac{7}{4}x^2 = 7 \Leftrightarrow x^2 = 4$

So $x = \pm 2$, by that $y = \frac{3}{2}x$, $y = \pm 3$.

So we have two points $(2, 3)$, $(-2, -3)$.

$f(2,3) = 14$ is the max., $f(-2,-3) = -14$ is the min.

Approach (ii): ① + 2 × ② $\Rightarrow 9 = 4\lambda y - \lambda y = 3\lambda y \Leftrightarrow y = \frac{3}{\lambda}$

plug $y = \frac{3}{\lambda}$ in ① $\Rightarrow x = \frac{2}{\lambda}$, So ③ becomes

$$\frac{4}{\lambda^2} + \frac{9}{\lambda^2} - \frac{6}{\lambda^2} = 7 \Rightarrow \lambda^2 = 1 \; (\lambda = \pm 1)$$

$\lambda = 1 \Rightarrow x = 2, y = 3$; $\lambda = -1 \Rightarrow x = -2, y = -3$.

5. (20 pts) Let C be the line segment from $(1,3)$ to $(4,7)$ which is parametrized by $\mathbf{r}(t) = \langle 3t+1, 4t+3 \rangle, 0 \le t \le 1$

(a) (8 pts) Compute the line integral $\int_C (2x+y)\, ds$.

(b) (12 pts) If $\mathbf{F} = \langle y, x-y \rangle$ is a vector field on C, find the circulation $\int_C \mathbf{F}\cdot\mathbf{T}\, ds$.

$\vec{r}\,'(t) = \langle 3, 4 \rangle$, $|\vec{r}\,'(t)| = \sqrt{3^2+4^2} = 5$

So (a) $\int_C (2x+y)\, ds = \int_0^1 \left(2(3t+1) + (4t+3)\right) |\vec{r}\,'(t)|\, dt$

$= 5\int_0^1 (6t+2+4t+3)\, dt = 5\int_0^1 (10t+5)\, dt$

$= 5(5t^2+5t)\Big|_0^1 = 5\cdot(5+5) = 50$

b) $\int_C \vec{F}\cdot\vec{T}\, ds = \int_0^1 \langle 4t+3, (3t+1)-(4t+3) \rangle \cdot \vec{r}\,'(t)\, dt$

$= \int_0^1 \langle 4t+3, -t-2 \rangle \cdot \langle 3, 4 \rangle\, dt$

$= \int_0^1 (12t+9-4t-8)\, dt$

$= \int_0^1 (8t+1)\, dt$

$= 4t^2+t\Big|_0^1$

$= 5$