Math 2153 **Name:**____________________________

Midterm 1 **OSU user name:**______________________

September 30, 2016 **Recitation Instructor:** ________________

The point value of each problem is indicated. To obtain full credit, you must have the correct answers along with the supporting work. Incorrect answers with work shown may receive partial credit, but **answers without supporting work may receive NO credit**, except True or False and fill the blank problems.

The exam consists of 6 problems starting on page 2 and ending on page 8. Make sure your exam is not missing any pages before you start.

Scientific calculators are allowed. Graphing calculators are NOT permitted.

Pros	Score
1	(22)
2	(12)
3	(20)
4	(18)
5	(18)
6	(10)
Total	(100)

1. (22 pts).

Given $P(1,1,1), Q(2,4,1), R(0,3,1+\sqrt{5})$. Denote θ the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(a) Fill the blank.

$\overrightarrow{PQ} = \underline{\langle 1, 3, 0 \rangle}$ $\quad\quad$ $\overrightarrow{PR} = \underline{\langle -1, 2, \sqrt{5} \rangle}$

$2\overrightarrow{PQ} - 3\overrightarrow{PR} = \underline{\langle 5, 0, -3\sqrt{5} \rangle}$ $\quad\quad$ $|\overrightarrow{QR}| = \underline{\sqrt{10}}$

$\theta = \underline{60^\circ \text{ or } \frac{\pi}{3}}$

$\text{scal}_{\overrightarrow{PR}}\overrightarrow{PQ} = \underline{\frac{5}{\sqrt{10}} \text{ or } \frac{\sqrt{10}}{2}}$

$\text{proj}_{\overrightarrow{PR}}\overrightarrow{PQ} = \underline{\langle -\frac{1}{2}, 1, \frac{\sqrt{5}}{2} \rangle}$

(b) Let W be the midpoint of the sigment PQ. Find the equation for the line passing through W and R.

$\left.\begin{array}{l} P = (1, 1, 1) \\ Q = (2, 4, 1) \end{array}\right\} \Rightarrow$ the midpoint $W = \left(\frac{1+2}{2}, \frac{1+4}{2}, \frac{1+1}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}, 1\right)$

a direction vector of the line is $\overrightarrow{WR} = \langle -\frac{3}{2}, \frac{1}{2}, \sqrt{5} \rangle$

so an equation of the line is $\vec{r}(t) = \langle \frac{3}{2}, \frac{5}{2}, 1 \rangle + t\langle -\frac{3}{2}, \frac{1}{2}, \sqrt{5} \rangle$

(c) Find the equation of the plane where the triangle ΔPQR lies in.

a normal vector of the plane is $\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ -1 & 2 & \sqrt{5} \end{vmatrix} = \langle 3\sqrt{5}, -\sqrt{5}, 5 \rangle$

a point in the plane is $P(1,1,1)$

So an equation of the plane is

$$3\sqrt{5}(x-1) + (-\sqrt{5})(y-1) + 5(z-1) = 0$$

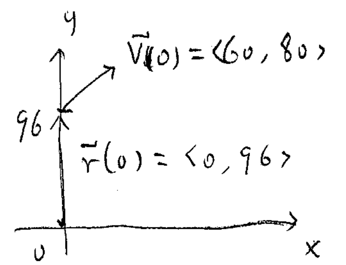
2. (12 pts)

If a golf ball is hit from 96 ft above the ground level, with an initial velocity of $\langle 60, 80\rangle$ ft/s. Given $g = 32$ ft/s^2 and assume there is only gravity.

(a) Find the position vector $\mathbf{r}(t)$ of the ball after t seconds.

(b) Find the time of flight and the horizontal distance it travels.

(c) Find the maximal height of the ball.



$\downarrow \vec{a}(t) = \langle 0, -g\rangle = \langle 0, -32\rangle$

(a) $\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \int 0\,dt, \int -32\,dt\rangle = \langle C_1, -32t + C_2\rangle$

$\vec{v}(0) = \langle 60, 80\rangle = \langle C_1, -32\cdot 0 + C_2\rangle = \langle C_1, C_2\rangle$

$C_1 = 60,\ C_2 = 80$

so $\vec{v}(t) = \langle 60, -32t + 80\rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \int 60\,dt, \int(-32t + 80)\,dt\rangle$
$= \langle 60t + C_3, -16t^2 + 80t + C_4\rangle$

$\vec{r}(0) = \langle 0, 96\rangle = \langle 60\cdot 0 + C_3, -16\cdot 0^2 + 80\cdot 0 + C_4\rangle = \langle C_3, C_4\rangle$

So $C_3 = 0$, $C_4 = 96$ and $\vec{r}(t) = \langle \underbrace{60t}_{s(t)}, \underbrace{-16t^2 + 80t + 96}_{h(t)}\rangle$

(b) $h(t) = 0 \Leftrightarrow 0 = t^2 - 5t - 6 = (t+1)(t-6) \Leftrightarrow t = 6$ or $t = -1$ (drop)

So the time of flight $= 6$ seconds, horizontal distane $= s(6) = 360$ ft

(c) max $h(t)$ happens at the time where y-component of $\vec{v}(t) = 0$,

i.e., $-32t + 80 = 0$, so $t = \frac{80}{32} = \frac{10}{4} = \frac{5}{2}$

So the max height $= h(\frac{5}{2}) = -16\cdot\frac{25}{4} + 80\cdot\frac{5}{2} + 96 = 196$ ft.

3. (20 pts)

Evaluate the following. If it is not well defined, state it and explain why. **Show your work.**

(a) $\frac{d}{dt}(\langle t, 2t, e^t\rangle \cdot \langle 0, t^2+1, t\rangle)$

$= \frac{d}{dt}\left(t \cdot 0 + 2t(t^2+1) + te^t \right)$

dot product is a number.

$= \frac{d}{dt}(2t^3 + 2t + te^t)$

$= 6t^2 + 2 + e^t + te^t$

the answer is a number.

(b) $\int_0^1 (\frac{1}{2t+1}\mathbf{i} + \sin(\pi t)\mathbf{j} + te^{t^2}\mathbf{k})\ dt.$

$= \left(\int_0^1 \frac{1}{2t+1} dt\right)\vec{i} + \left(\int_0^1 \sin(\pi t) dt\right)\vec{j} + \left(\int_0^1 te^{t^2} dt\right)\vec{k}$

$= \left(\frac{1}{2}\ln(2t+1)\Big|_0^1\right)\vec{i} + \left(\frac{-1}{\pi}\cos\pi t\Big|_0^1\right)\vec{j} + \left(\frac{1}{2}e^{t^2}\Big|_0^1\right)\vec{k}$

$= \left(\frac{1}{2}\ln 3 - \frac{1}{2}\ln 1\right)\vec{i} + \left(\frac{1}{\pi} - \left(-\frac{1}{\pi}\right)\right)\vec{j} + \left(\frac{1}{2}e^1 - \frac{1}{2}e^0\right)\vec{k}$

$= \left(\frac{1}{2}\ln 3\right)\vec{i} + \frac{2}{\pi}\vec{j} + \left(\frac{1}{2}e - \frac{1}{2}\right)\vec{k}$

the answer is a vector!

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+x-xy-y}{x-y}$. (plug-in of $(0,0) \longrightarrow$ "$\frac{0}{0}$")

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)+(x-y)}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x+1)(x-y)}{x-y}$$

$$= \lim_{(x,y)\to(0,0)} (x+1) = 0+1 = 1$$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2}$. (plug-in of $(0,0) \longrightarrow$ "$\frac{0}{0}$")

(can't simplify)

(try the two-path test)

"x^2+y^2" in the bottom indicates that we can try $y=mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2-xy+y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-xmx+m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{1-m+m^2}{1+m^2} = \frac{1-m+m^2}{1+m^2} = \begin{cases} 1 & \text{if } m=0 \\ \frac{1}{2} & \text{if } m=1 \end{cases}$$

So by the two-path test, the original limit does not exist.

4. (18 pts)

Given the ellipse with equation $\dfrac{x^2}{4} + y^2 = 1$.

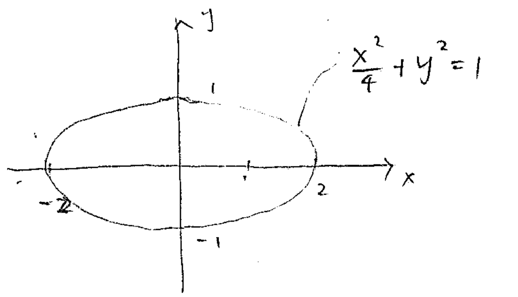
(a) Briefly sketch the graph of the ellipse in xy-coordinate.

(b) Let $P = (2, 0), Q = (0, 1)$. Compute the curvature κ of the ellipse at P and Q.

Which point has greater curvature?

(c) Find the torsion τ of the ellipse. (Hint: let $x = 2\cos t, y = \sin t$).

a)



b) The curve is in fact (by the hint) $\vec{r}(t) = \langle 2\cos t, \sin t \rangle$

$P = (2,0) \leftrightarrow t = 0$, $\quad Q = (0,1) \longleftrightarrow t = \frac{\pi}{2}$

$$\kappa(t) = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}, \qquad \vec{v} = \vec{r}'(t) = \langle -2\sin t, \cos t, 0 \rangle$$
$$\vec{a} = \vec{v}' = \langle -2\cos t, -\sin t, 0 \rangle$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & \cos t & 0 \\ -2\cos t & -\sin t & 0 \end{vmatrix} = (2\sin^2 t + 2\cos^2 t)\,\vec{k} = \langle 0, 0, 2 \rangle$$

$$|\vec{v} \times \vec{a}| = \sqrt{0^2 + 0^2 + 2^2} = 2, \qquad |\vec{v}|^3 = \sqrt{(2\sin t)^2 + \cos^2 t} = \sqrt{3\sin^2 t + 1}$$

So $$\kappa(t) = \frac{2}{(3\sin^2 t + 1)^{\frac{3}{2}}}, \qquad \kappa_P = \kappa(0) = 2, \quad \kappa_Q = \kappa\left(\tfrac{\pi}{2}\right) = \frac{2}{8} = \frac{1}{4}$$

$\kappa_P > \kappa_Q$ (which is also clear from the picture)

c) Since the curve is in the xy-plane, the torsion is zero.

5. (18 pts)

(a) Let $f(x, y) = \sin(e^{x-y})$. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

$$\frac{\partial f}{\partial x} = \cos(e^{x-y}) \frac{\partial}{\partial x}(e^{x-y}) = \cos(e^{x-y}) \cdot e^{x-y}$$

$$\frac{\partial f}{\partial y} = \cos(e^{x-y}) \frac{\partial}{\partial y}(e^{x-y}) = \cos(e^{x-y}) \cdot (-e^{x-y})$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\cos(e^{x-y})\, e^{x-y}\right) = -\frac{\partial}{\partial x}\left(\cos(e^{x-y})\, e^{x-y}\right)$$

$$= -\left(-\sin(e^{x-y})\, e^{x-y} \cdot e^{x-y} + \cos(e^{x-y}) \cdot e^{x-y}\right)$$

$$= \sin(e^{x-y})\, e^{2(x-y)} - \cos(e^{x-y})\, e^{x-y}$$

(b) If $z = e^x \sqrt{y^2 + 3}$ and $x = rs,\ y = s - 2r$, find $\frac{\partial z}{\partial r}|_{(r,s)=(0,1)}$.

By the chain rule,

① $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial x} = e^x \sqrt{y^2+3}, \quad \frac{\partial x}{\partial r} = s$$

$$\frac{\partial z}{\partial y} = e^x \frac{1}{2}(y^2+3)^{-\frac{1}{2}} \cdot 2y = e^x \frac{y}{\sqrt{y^2+3}}$$

$$\frac{\partial y}{\partial r} = -2$$

$r=0, s=1 \Rightarrow x = rs = 0,\ y = s - 2r = 1$

So plugging these in ① gives

$$\frac{\partial z}{\partial r}\bigg|_{(r,s)=(0,1)} = e^0\sqrt{4} \cdot 1 + e^0 \cdot \frac{-2}{\sqrt{4}} = 2 - 1 = 1$$

alternatively,

$$z = e^{rs}\sqrt{(s-2r)^2+3}$$

$$\frac{\partial z}{\partial r} = s e^{rs}\sqrt{(s-2r)^2+3} + e^{rs} \frac{2(s-2r)}{2\sqrt{(s-2r)^2+3}}$$

plugging in $r=0, s=1$ gives the same answer.

6. (10 pts)

Determine if the following statements are True of False by circling the letter T for True or F for False. You do not need to show any explanations for this problems.

F (a) T / F $\qquad \vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$

u and **v** are two nonzero vectors. If **u** is parallel to **v**, then $\mathbf{u} \cdot \mathbf{v} = 0$.

T (b) T / F

Given $f(x, y) = x^2 - 2y + 1$, the level curve $f(x, y) = 0$ is a parabola passing through $P(1, 1)$. level curve in question is $0 = x^2 - 2y + 1$, i.e., $y = \frac{x^2}{2} + \frac{1}{2}$, which is a parabola w/ (1,1) on it.

T (c) T / F $\quad 4 = |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$, differentiating both sides by product rule gives $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ $\Rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$

If $|\mathbf{r}(t)| = 2$ for any t, then the vector $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

F (d) T / F

The curve given by $\mathbf{r}(t) = \langle\sqrt{3}t, \sin t, \cos t\rangle, t \geq 0$ uses arc length as the parameter.

$|\vec{r}\,'(t)| = \sqrt{(\sqrt{3})^2 + (\cos t)^2 + (-\sin t)^2} = \sqrt{3+1} = 2 \neq 1$

so $\vec{r}(t)$ does not use arc length as the parameter.

F (e) T / F

$$f(x, y) = \begin{cases} \dfrac{\tan(x^2 + y^2)}{x^2 + y^2}, (x, y) \neq (0, 0) \\ 1, (x, y) = (0, 0) \end{cases}$$

is not continuous and not differentiable at $(0, 0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2} \underset{u=x^2+y^2}{=} \lim_{u\to 0} \frac{\tan u}{u} = \lim_{u\to 0} \frac{\sin u}{u} \cdot \frac{1}{\cos u} = 1 = f(0,0)$$

So f is continuous at $(0, 0)$.

Formulas

Velocity: $\mathbf{v} = \mathbf{r}'(t)$

Acceleration: $\mathbf{a} = \mathbf{r}''(t)$

Unit tange vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, provided $d\mathbf{T}/dt \neq 0$

Curvature: $\kappa = |d\mathbf{T}/ds| = \dfrac{1}{|\mathbf{v}|}|\dfrac{d\mathbf{T}}{dt}| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Unit binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \dfrac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}$