Math 1172 - Autumn 2015 Name: ____________________

Quiz 4 - Take Home Recitation Instructor: ____________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit.

Problem 1 [4 points] Find an appropriate trigonometric substitution of one of the forms $x = C\sin\theta$, $x = C\sec\theta$, or $x = C\tan\theta$ to simplify the integral given below. Simplfy the integrand as much as possible and evaluate.

$$4 + x^2 = 2^2 + x^2 \Rightarrow \text{use } x = 2\tan\theta$$

$$\Rightarrow dx = 2\sec^2\theta\, d\theta$$

$$\mathcal{I} = \int \frac{x^4}{(4+x^2)^{3/2}}\,dx \qquad (4+x^2)^{3/2} = (4+4\tan^2\theta)^{3/2} = 4^{3/2}(1+\tan^2\theta)^{3/2}$$

$$= 8(\sec^2\theta)^{3/2} = 8\cdot\sec^3\theta.$$

$$x^4 = 16\tan^4\theta$$

$$\text{So } \mathcal{I} = \int \frac{x^4}{(4+x^2)^{3/2}}\,dx = \int \frac{16\tan^4\theta\cdot 2\sec^2\theta\, d\theta}{8\sec^3\theta} = 4\int \frac{\tan^4\theta}{\sec\theta}\,d\theta$$

$$= 4\int \frac{(\sec^2\theta-1)^2}{\sec\theta}\,d\theta = 4\int \frac{\sec^4\theta - 2\sec^2\theta + 1}{\sec\theta}\,d\theta.$$

$$= 4\int \sec^3\theta - 2\sec\theta + \cos\theta\, d\theta$$

$$= 4\underbrace{\int \sec^3\theta\, d\theta}_{①} - 8\underbrace{\int \sec\theta\, d\theta}_{②} + 4\underbrace{\int \cos\theta\, d\theta}_{③}$$

$$③ = \sin\theta + C_3$$

$$② = \int \sec\theta \cdot \frac{\tan\theta + \sec\theta}{\tan\theta+\sec\theta}\,d\theta = \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\tan\theta+\sec\theta}\,d\theta = \int \frac{(\tan\theta+\sec\theta)'}{\tan\theta+\sec\theta}\,d\theta =$$

$$= \ln|\tan\theta + \sec\theta| + C_2$$

For ①, we use the reduction formula

$$\int \sec^n x\, dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx \quad (\text{with } n \neq 1)$$

for $n=3$. So

$$① = \int \sec^3\theta\, d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int\sec\theta\, d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln|\tan\theta+\sec\theta| + C_1$$

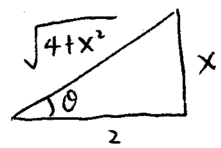
So the original integral

$$\mathcal{I} = 2\sec\theta\tan\theta - 6\ln|\tan\theta+\sec\theta| + 4\sin\theta + C$$

(see next page)

Now we need to substitute back to get a function of x.

$x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow$ the following figure



So $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4+x^2}}{2}$ and $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{4+x^2}}$.

Hence

$$
\begin{aligned}
I &= 2\sec\theta\tan\theta - 6\ln|\tan\theta + \sec\theta| + 4\sin\theta + C \\
&= 2\cdot\frac{\sqrt{4+x^2}}{2}\cdot\frac{x}{2} - 6\ln\left|\frac{x}{2} + \frac{\sqrt{4+x^2}}{2}\right| + 4\cdot\frac{x}{\sqrt{4+x^2}} + C \\
&= \frac{x\sqrt{4+x^2}}{2} - 6\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right| + \frac{4x}{\sqrt{4+x^2}} + C.
\end{aligned}
$$