Some statistical models and corresponding SAS software code (Normal [left] or binomial [right])

Design	LMM (Normal cond. dist.)	SAS code (LMM; normal)	GLMM (Binomial cond. dist.)	SAS code (GLMM; binomial)

Completely randomized (CR), one factor	$\mu_i = \eta_i = \theta + \alpha_i$
	$Y_{ij} \sim N(\mu_i, \sigma^2)$
	
	j is an index for the j-th experimental unit of the treatment (such as the plot)
		`class alpha ;`
		`model y = alpha;`
			Naïve GLMM
			$\text{logit}(p_i) = \eta_i = \theta + \alpha_i$
			$Y_i \sim Bin(p_i, n)$
				`class alpha ;`
				`model y/n = alpha`
				`  / dist=bin link=logit;`

			Cond. GLMM (w/ unit effect)
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + v_{ij}$
			$Y_{ij} \mid v_{ij} \sim Bin(p_{ij}, n)$
			$v_{ij} \sim N(0,\sigma_v^2)$ experimental unit (*j*) within treatment
				`class alpha ;`
				`model y/n = alpha`
				`  / dist=bin link=logit;`
				`random plot(alpha);`
				
				plot is a code to identify each unique experimental unit receiving a treatment

			Quasi-likelihood
			$\text{logit}(p_i) = \eta_i = \theta + \alpha_i$
			$Y_i \sim quasi\text{-}Bin(p_i, n; \phi)$
				`class alpha ;`
				`model y/n = alpha`
				`  / dist=bin link=logit;`
				`random _residual_;`

Randomized complete block (RCB), one factor	$\mu_{ij} = \eta_{ij} = \theta + \alpha_i + b_j$
	$Y_{ij} \mid b_j \sim N(\mu_{ij}, \sigma_e^2)$
	$b_j \sim N(0,\sigma_b^2)$
		`class alpha block;`
		`model y = alpha;`
		`random block;`
			Naïve GLMM
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + b_j$
			$Y_{ij} \mid b_j \sim Bin(p_{ij}, n)$
			$b_j \sim N(0,\sigma_b^2)$
				`class alpha block;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block;`

			Cond. GLMM (w/ unit effect)
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + b_j + v_{ij}$
			$Y_{ij} \mid b_j, v_{ij} \sim Bin(p_{ij}, n)$
			$b_j \sim N(0,\sigma_b^2)$
			$v_{ij} \sim N(0,\sigma_v^2)$
				`class alpha block;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block block*alpha;`

			Quasi-likelihood
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + b_j$
			$Y_{ij} \mid b_j \sim quasi\text{-}Bin(p_{ij}, n; \phi)$
			$b_j \sim N(0,\sigma_b^2)$
				`class alpha block;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block;`
				`random _residual_;`

RCB, two factors	$\mu_{ijk} = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k$
	$Y_{ijk} \mid b_j \sim N(\mu_{ijk}, \sigma_e^2)$
		`class alpha beta block;`
		`model y = alpha|beta;`
		`random block;`
			Naïve GLMM
			$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k$
			$Y_{ijk} \mid b_k \sim Bin(p_{ijk}, n)$
				`class alpha beta block;`
				`model y/n = alpha|beta`
				`   / dist=bin link=logit;`
				`random block;`

<table>
<tr><td></td><td>$b_j \sim N(0,\sigma_b^2)$</td><td>--or, equivalently--
<code>random int / sub=block;</code></td><td>$b_k \sim N(0,\sigma_b^2)$</td><td></td></tr>
<tr><td></td><td></td><td></td><td>Cond. GLMM (w/ unit effect)
$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k + v_{ijk}$
$Y_{ijk} \mid b_k, v_{ijk} \sim Bin(p_{ijk}, n)$
$b_k \sim N(0,\sigma_b^2)$
$v_{ijk} \sim N(0,\sigma_v^2)$</td><td><code>class alpha beta block;</code>
<code>model y/n = alpha|beta</code>
<code>/ dist=bin link=logit;</code>
<code>random block</code>
<code>block*alpha*beta;</code></td></tr>
<tr><td></td><td></td><td></td><td>Quasi-likelihood
$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k$
$Y_{ijk} \mid b_k \sim quasi\text{-}Bin(p_{ijk}, n; \phi)$
$b_k \sim N(0,\sigma_b^2)$</td><td><code>class alpha beta block;</code>
<code>model y/n = alpha|beta</code>
<code>/ dist=bin link=logit;</code>
<code>random block;</code>
<code>random _residual_;</code></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td><u>RCB, Split plot</u>
Whole-plot (α)
Sub-plot (β)</td><td>$\mu_{ijk} = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k + d_{ik}$
$Y_{ijk} \mid b_k, d_{ik} \sim N(\mu_{ijk}, \sigma_e^2)$ Conditional dist.
$b_k \sim N(0,\sigma_b^2)$ block
$d_{ik} \sim N(0, \sigma_d^2)$ whole-plot
(=b x α)</td><td><code>class alpha beta block;</code>
<code>model y = alpha|beta;</code>
<code>random block</code>
<code>block*alpha;</code>
--or, equivalently--
<code>random int alpha</code>
<code>/ sub=block;</code></td><td>Naïve GLMM
$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k + d_{ik}$
$Y_{ijk} \mid b_k, d_{ik} \sim Bin(p_{ijk}, n)$
$b_k \sim N(0,\sigma_b^2)$
$d_{ik} \sim N(0, \sigma_d^2)$</td><td><code>class alpha beta block;</code>
<code>model y = alpha|beta</code>
<code>/ dist=bin link=logit;</code>
<code>random block</code>
<code>block*alpha;</code>
--or, equivalently--
<code>random int alpha</code>
<code>/ sub=block;</code></td></tr>
<tr><td></td><td></td><td></td><td>Cond. GLMM (w/ unit effect)
$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k + d_{ik} + v_{ijk}$
$Y_{ijk} \mid b_k, d_{ik}, v_{ijk} \sim Bin(p_{ijk}, n)$
$b_k \sim N(0,\sigma_b^2)$
$d_{ik} \sim N(0, \sigma_d^2)$
$v_{ijk} \sim N(0,\sigma_v^2)$</td><td><code>class alpha beta block;</code>
<code>model y = alpha|beta</code>
<code>/ dist=bin link=logit;</code>
<code>random block</code>
<code>block*alpha</code>
<code>block*alpha*beta;</code>
--or, equivalently--
<code>random int alpha alpha*beta</code>
<code>/ sub=block;</code></td></tr>
<tr><td></td><td></td><td></td><td>Quasi-likelihood
$\text{logit}(p_{ijk}) = \eta_{ijk} = \theta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + b_k + d_{ik}$
$Y_{ijk} \mid b_k, d_{ik} \sim quasi\text{-}Bin(p_{ijk}, n; \phi)$</td><td><code>class alpha beta block;</code>
<code>model y = alpha|beta</code>
<code>/ dist=bin link=logit;</code>
<code>random block</code>
<code>block*alpha;</code>
<code>random _residual_;</code></td></tr>
</table>

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<tr><td></td><td></td><td></td><td>$b_k \sim N(0,\sigma_b^2)$
$d_{ik} \sim N(0, \sigma_d^2)$</td><td>--or, equivalently--
random int alpha
/ sub=block;
random _residual_;</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td><u>RCB, split-split plot</u>
Whole-plot (α)
Sub-plot (β)
Sub-sub plot (γ)</td><td>$\mu_{ijkl} = \eta_{ijkl} = \theta + \alpha_i + \beta_j$
$+ \gamma_l + (\alpha\beta)_{ij} + (\alpha\gamma)_{il}$
$+ (\beta\gamma)_{jl} + (\alpha\beta\gamma)_{ijl}$
$+ b_k + d_{ik} + f_{ijk}$

$Y_{ijkl} \mid b_k, d_{ik}, f_{ijk} \sim N(\mu_{ijkl}, \sigma_e^2)$
$b_k \sim N(0,\sigma_b^2)$ block
$d_{ik} \sim N(0, \sigma_d^2)$ whole-plot (=b x α)
$f_{ijk} \sim N(0, \sigma_f^2)$ sub-plot (=bxαxβ)</td><td>class alpha beta gamma
block;
model y=
alpha|beta|gamma ;
random block block*alpha
block*alpha*beta;

--or, equivalently--
random int alpha
alpha*beta /sub=block;</td><td>Naïve GLMM
$\mathrm{logit}(p_{ijkl}) = \eta_{ijkl} = \theta + \alpha_i + \beta_j$
$+ \gamma_l + (\alpha\beta)_{ij} + (\alpha\gamma)_{il}$
$+ (\beta\gamma)_{jl} + (\alpha\beta\gamma)_{ijl}$
$+ b_k + d_{ik} + f_{ijk}$

$Y_{ijkl} \mid b_k, d_{ik}, f_{ijk} \sim Bin(p_{ijkl}, n)$
$b_k \sim N(0,\sigma_b^2)$ block
$d_{ik} \sim N(0, \sigma_d^2)$ whole-plot (=b x α)
$f_{ijk} \sim N(0, \sigma_f^2)$ sub-plot (=b x α x β)</td><td>class alpha beta gamma
block;
model y=
alpha|beta|gamma /
dist=bin link=logit;
random block block*alpha
block*alpha*beta;

--or, equivalently--
random int alpha
alpha*beta /sub=block;</td></tr>
<tr><td></td><td></td><td></td><td>Cond. GLMM (w/ unit effect)
$\mathrm{logit}(p_{ijkl}) = \eta_{ijkl} = \theta + \alpha_i + \beta_j$
$+ \gamma_l + (\alpha\beta)_{ij} + (\alpha\gamma)_{il}$
$+ (\beta\gamma)_{jl} + (\alpha\beta\gamma)_{ijl}$
$+ b_k + d_{ik} + f_{ijk} + v_{ijkl}$

$Y_{ijkl} \mid b_k, d_{ik}, f_{ijk}, v_{ijkl} \sim Bin(p_{ijkl}, n)$
$b_k \sim N(0,\sigma_b^2)$
$d_{ik} \sim N(0, \sigma_d^2)$
$f_{ijk} \sim N(0, \sigma_f^2)$
$v_{ijkl} \sim N(0,\sigma_v^2)$</td><td>class alpha beta gamma
block;
model y=
alpha|beta|gamma /
dist=bin link=logit;
random block block*alpha
block*alpha*beta
block*alpha*beta*gamma;

--or, equivalently--
random int alpha
alpha*beta
alpha*beta*gamma
/ sub=block;</td></tr>
<tr><td></td><td></td><td></td><td>Quasi-likelihood
$\mathrm{logit}(p_{ijkl}) = \eta_{ijkl} = \theta + \alpha_i + \beta_j$
$+ \gamma_l + (\alpha\beta)_{ij} + (\alpha\gamma)_{il}$
$+ (\beta\gamma)_{jl} + (\alpha\beta\gamma)_{ijl}$
$+ b_k + d_{ik} + f_{ijk}$

$Y_{ijkl} \mid b_k, d_{ik}, f_{ijk} \sim$
$quasi\text{-}Bin(p_{ijkl}, n; \phi)$</td><td>class alpha beta gamma
block;
model y=
alpha|beta|gamma /
dist=bin link=logit;
random block block*alpha
block*alpha*beta;
random _residual_ ;

--or, equivalently--</td></tr>
</table>

			$b_k \sim N(0,\sigma_b^2)$
			$d_{ik} \sim N(0, \sigma_d^2)$
			$f_{ijk} \sim N(0, \sigma_f^2)$
				`random int alpha`
				`  alpha*beta /sub=block;`
				`random _residual_ ;`

RCB, one factor, with **sub-sampling**
 	
Normal: Linear predictor for *ij* treatment-block combination, with cond. dist. of *Y* for *ijk* treat-block-sampling unit (su) combination
	$\mu_{ij} = \eta_{ij} = \theta + \alpha_i + b_j + v_{ij}$
	$Y_{ijk} \mid b_j, v_{ij} \sim N(\mu_{ij}, \sigma_e^2)$
	$b_j \sim N(0,\sigma_b^2)$
	$v_{ij} \sim N(0, \sigma_v^2)$
		`class alpha block;`
		`model y = alpha;`
		`random block`
		`   block*alpha;`
			Naïve GLMM
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + b_j + v_{ij}$
			$Y_{ijk} \mid b_j, v_{ij} \sim Bin(p_{ij}, n)$
			$b_j \sim N(0,\sigma_b^2)$
			$v_{ij} \sim N(0, \sigma_v^2)$
				`class alpha block;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block block*alpha;`
				
				--or, equivalently--
				`random int alpha /`
				`       sub=block;`

Binomial (with unit-level effect): Linear predictor *and* cond. dist. of *Y* for *ijk* treat-block-sampling unit (su) combination			**Cond. GLMM (w/ unit effect)**
			$\text{logit}(p_{ijk}) = \eta_{ij} = \theta + \alpha_i$
			$+ b_j + v_{ij} + u_{ijk}$
			
			$Y_{ijk} \mid b_j, v_{ij}, u_{ijk} \sim Bin(p_{ijk}, n)$
			$b_j \sim N(0,\sigma_b^2)$
			$v_{ij} \sim N(0, \sigma_v^2)$
			$u_{ijk} \sim N(0, \sigma_u^2)$
				`class alpha block su;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block block*alpha`
				`   block*alpha*su;`
				--or, equivalently--
				`random int alpha`
				`   alpha*su;/ sub=block ;`

Quasi-likelihood: Linear predictor for *ij* treatment-block combination, with cond. quasi-dist. of *Y* for *ijk* treat-block-sampling unit (su) combination			**Quasi-likelihood**
			$\text{logit}(p_{ij}) = \eta_{ij} = \theta + \alpha_i + b_j + v_{ij}$
			$Y_{ijk} \mid b_j, v_{ij} \sim quasi\text{-}Bin(p_{ij}, n; \phi)$
			$b_j \sim N(0,\sigma_b^2)$
			$v_{ij} \sim N(0, \sigma_v^2)$
				`class alpha block;`
				`model y/n = alpha`
				`   / dist=bin link=logit;`
				`random block block*alpha;`
				`random _residual_;`
				
				--or, equivalently--
				`random int alpha /`
				`       sub=block;`
				`random _residual_;`

Unless indicated otherwise, Greek letters indicate fixed-effect terms (treatment or factor A [alpha: α], factor B [beta: β], etc.), and Roman letters indicate random effects (*b* is the random block effect, *d* is the whole-plot effect, etc.).

For other distributions in the exponential family, change `dist=` and `link=` options. For example, for counts, use `dist=Poisson link=log`.

The quasi-likelihood approach is appropriate for binomial or Poisson conditional distributions (those without a so-called free scale parameter). Note that *Bin(•,n; ϕ)* is a customized notation for a quasi-distribution, an overdispersed “binomial” in this case (where ϕ is the overdispersion scale parameter).