Problem 1

[20 pts] Use a suitable change of variables to evaluate the integral \( \iint_R \frac{x-2y}{2x-y} \, dA \)

where \( R \) is the region bounded by the graphs of \( 2x - y = 1, \, 2x - y = 4, \, x - 2y = 4 \), and \( x - 2y = 8 \).

\( R \) is bounded by two pairs of parallel lines. This suggests the following change of variables:

\[
\begin{align*}
\{ & u = 2x - y \quad \circ \cr & v = x - 2y \quad \circ \cr \end{align*}
\]

Then the new region \( R' \) is \( \{(u,v) \mid 1 \leq u \leq 4, \, 4 \leq v \leq 8\} \).

To find \( J(u,v) \), we need to solve \( \circ \) and \( \circ \) for \( x \) and \( y \)

\( \circ - 2 \cdot \circ \) gives \( u-2v = 3y \) and \( y = \frac{1}{3}u - \frac{2}{3}v \).

Plug it in \( \circ \) to get \( x = v+2y = \frac{1}{3}u + \frac{1}{3}v \). So

\[
\begin{align*}
\{ & x = \frac{1}{3}u + \frac{1}{3}v \cr & y = \frac{1}{3}u - \frac{2}{3}v \cr \end{align*}
\]

And

\[
J(u,v) = \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}
\end{array} \right| = \left| \begin{array}{cc}
\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{2}{3}
\end{array} \right| = -\frac{1}{3} \cdot \frac{2}{3} - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{3}
\]

\[
|J(u,v)| = \left| -\frac{1}{3} \right| = \frac{1}{3}.
\]

Then

\[
\iint_R \frac{x-2y}{2x-y} \, dA = \iint_{R'} \frac{v}{u} |J(u,v)| \, dA = \frac{1}{3} \int_4^8 \int_1^4 \frac{v}{u} \, du \, dv
\]

\[
= \frac{1}{3} \int_4^8 \left. (v \cdot \ln|u|) \right|_{u=1}^{u=4} \, dv = \frac{1}{3} \int_4^8 ((\ln 4) \cdot v) \, dv
\]

\[
= \frac{\ln 4}{3} \cdot \frac{v^2}{2} \bigg|_{v=4}^{v=8} = \frac{\ln 4}{6} (8^2 - 4^2) = 8 \ln 4 = 16 \ln 2.
\]
Consider the vector field \( \mathbf{F}(x, y, z) = 6xz \mathbf{i} + (3y^2 + 1) \mathbf{j} + (3x^2 - 2z + y^3) \mathbf{k} \).

a) [6 pts] Show that \( \mathbf{F} \) is a conservative vector field on \( \mathbb{R}^3 \).

\[
\begin{align*}
\frac{\partial R}{\partial y} &= 3y^2, & \frac{\partial Q}{\partial x} &= \frac{\partial P}{\partial z} \\
\frac{\partial R}{\partial x} &= 6x, & \frac{\partial Q}{\partial y} &= \frac{\partial P}{\partial x} \Rightarrow \mathbf{F} \text{ is a conservative field.}
\end{align*}
\]

Domain \( \mathbb{R}^3 \) is simply connected.

b) [12 pts] Find a potential function \( f \) for \( \mathbf{F} \) such that \( \nabla f = \mathbf{F} \).

\[
f = \int P \, dx = \int 6xz \, dx = 3x^2z + C(y, z),
\]

\[
f_y = Q \Rightarrow \frac{\partial}{\partial y} \left( 3x^2z + C(y, z) \right) = \frac{\partial}{\partial y} C(y, z) = 3y^2z + 1.
\]

So \( C(y, z) = \int (3y^2z + 1) \, dy = y^3z + y + D(z) \) and \( f = 3x^2z + y^3z + y + D(z) \). Then \( f_z = R \Rightarrow \frac{\partial}{\partial z} \left( 3x^2z + y^3z + y + D(z) \right) = 3x^2y^3 + D'(z) = 3x^2 - 2z + y^3 \)

and \( D'(z) = -2z \) and \( D(z) = \int -2z \, dz = -z^2 + \text{constant} \).

Hence \( f = 3x^2z + y^3z + y - z^2 + \text{constant} \).

c) [2 pts] Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is any oriented curve from \((1,1,0)\) to \((1,-1,0)\). Since \( \mathbf{F} \) is a conservative vector field with a potential function \( f = 3x^2z + y^3z + y - z^2 \), the fundamental theorem for line integrals (FTLI) says

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,-1,0) - f(1,1,0) = -1 - 1 = -2.
\]
Evaluate the line integral.

a) [6 pts] \( \int_C xy \, ds \) where \( C \) is the line segment joining \((-1, 2)\) to \((2, 0)\). 

First, find the parametrization of the line segment:
\[ \vec{r}(t) = \langle -1, 2 \rangle + t \langle 3, -2 \rangle = \langle 3t - 1, -2t + 2 \rangle, \quad 0 \leq t \leq 1. \]

Then \( \vec{r}'(t) = \langle 3(t - 1), -(2t + 2) \rangle = \langle 3, -2 \rangle \) and \( |\vec{r}'(t)| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \)

So \[ \int_C xy \, ds = \int_0^1 \langle 3(t - 1), -(2t + 2) \rangle \cdot \sqrt{13} \, dt = \int_0^1 (-6t^2 + 8t - 2) \, dt \]
\[ = -2t^3 + 4t^2 - 2t \bigg|_0^1 = 0 \]

b) [6 pts] \( \int_C x^2 y \, ds \) where \( C \) is the upper half of the unit circle \( x^2 + y^2 = 1 \).

A parametrization of the upper half of the unit circle is
\[ \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \pi. \]

Since \( \vec{r}'(t) = \langle -\sin t, \cos t \rangle \) and \( |\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \)

\[ \int_C x^2 y \, ds = \int_0^\pi \cos^2 t \sin t \cdot 1 \, dt = \int_0^\pi (-u^3) \, du = \left[-\frac{u^4}{4}\right]_0^\pi = -\left(\frac{-1}{4}\right)^4 - \left(\frac{1}{4}\right)^4 = \frac{2}{3} \]

c) [8 pts] \( \int_C -y \, dx + (x + y) \, dy \) where \( C \) is the unit circle \( x^2 + y^2 = 1 \) oriented counterclockwise. A parametrization of the curve in question is
\[ \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi. \] We have \( \vec{r}'(t) = \langle -\sin t, \cos t \rangle \)

so \( x'(t) = -\sin t, \quad y'(t) = \cos t. \) Then
\[ \int_C -y \, dx + (x + y) \, dy = \int_0^{2\pi} (-\cos t)(-\sin t) \, dt + (\cos t + \sin t) \cos t \, dt \]
\[ = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t + \sin t \cos t\right) \, dt = \int_0^{2\pi} 1 + \frac{1}{2} \sin(2t) \, dt \]
\[ = t - \frac{1}{4} \cos(2t) \bigg|_0^{2\pi} = (2\pi - \frac{1}{4}) - (0 - \frac{1}{4}) = 2\pi \]
Problem 4

[20 pts] Find the volume of the solid region in the first octant bounded by the graphs $z = 1 - x^2$ and $z = y$.

Sketch the solid.

$z = 1 \Rightarrow y = 1 \& x = 0$.

So $(0, 1, 1)$ is a point of intersection.

Proj solid onto $xz$ plane to get $R$ (shaded)

$$Vol = \int \int \left( \int \frac{1}{G(x,z)} \right) dx$$

$$= \int_0^1 \int_0^{1-x^2} \frac{1}{1-x^2} \, dy \, dz \, dx$$

$$= \int_0^1 \int_0^{1-x^2} \frac{1}{z} \, dz \, dx$$

$$= \int_0^1 \left[ \frac{z}{2} \right]_{z=0}^{z=1-x^2} \, dx$$

$$= \int_0^1 \frac{1}{2} (1-x^2)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 (1-x^2)^2 \, dx$$

$$= \frac{1}{2} \left[ \frac{x^3}{3} + \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{3} + \frac{1}{10} = \frac{15}{30} - \frac{10}{30} + \frac{3}{30} = \frac{8}{30} = \frac{4}{15}$$
Problem 5

[20 pts] Use spherical coordinates to find the volume of the region outside the cone \( \phi = \pi/6 \) and inside the sphere of radius 3 centered at \((0, 0, 3)\).

Sketch the solid:

\[
\text{sphere equation } p = 2 \cdot 3 \cos \phi = 6 \cos \phi
\]

Using spherical coordinates,

\[
\text{Vol} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^6 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \left( \frac{1}{3} \rho^3 \sin \phi \right) \bigg|_{\rho=6}^{\rho=0} \, d\rho \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \frac{6^3}{3} \cos \phi \sin \phi \, d\phi \, d\theta
\]

\[
= 72 \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta
\]

\[
= 72 \int_0^{2\pi} \int_0^{\pi/2} u^3 (-du) \, d\theta = 72 \int_0^{2\pi} \left( -\frac{u^4}{4} \bigg|_{u=\sin \theta}^{u=1} \right) \, d\theta
\]

\[
= 72 \int_0^{2\pi} \left( 0 - (-\frac{9}{4}) \right) \, d\theta = \frac{72 \cdot 9 \cdot 2\pi}{64} = \frac{81}{4} \pi
\]