ECE2000 - Homework 8 Solutions

Problems 4.1, 4.3, 4.4, 4.8, 4.10, 4.11, 4.13

4.1) a) Period of x(t):  \( T_0 = \frac{2\pi}{\omega_0} = 2.273 \text{ ms/cycle} \)

\[ T_s = 0.1 \text{ ms/sample} \quad \rightarrow \quad T_0/T_s = 22.73 \text{ samples/cycle} \]

b) Remember that \( x[n] = x(nT_s) \), plugging \( t = nT_s \) into \( x(t) \):

\[ x(nT_s) = 10\cos(880\pi nT_s + \phi) \]

\[ = 10\cos(880\pi nT_s + 2\pi n + \phi) \]

\[ = 10\cos((880\pi + 2\pi/Ts)nT_s + \phi) \]

\[ \omega_0 = 880\pi + 2\pi/Ts = 20,880\pi \]

c) Note that \( \omega_0 \) is in rad/sec while \( F_s \) is in Hz. Be sure to convert one of these values so that both have the same units.

\[ F_0 = 10,440 \text{ cycles/sec} \quad F_s = 1/T_0 = 10,000 \text{ samples/sec} \]

\[ F_s/F_0 = 10000 \text{ Hz}/10440 \text{ Hz} = 0.978 \text{ samples/cycle} \]

4.3) \( x[n] = 2.2\cos(0.3\pi n - \pi/3) = 2.2\cos(0.3\pi n + 2\pi k^*n - \pi/3) \)

\[ = 2.2\cos([0.3\pi + 2\pi k]n - \pi/3) \]

*Since we can use any integer number (k) of 2\pi and get the same result*

\[ \tilde{\omega} = 0.3\pi + 2\pi k \quad F_0 = \frac{\tilde{\omega}}{2\pi} F_s = (0.15 + k)6000 \text{ Hz} \]

<table>
<thead>
<tr>
<th>k</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0 )</td>
<td>-11100</td>
<td>-5100</td>
<td>900</td>
<td>6900</td>
<td>12900</td>
</tr>
</tbody>
</table>

Choosing 5100, 900, and 6900 since they are all less than 8000 Hz

\[ x_{-1}(t) = 2.2\cos(2\pi(-5100)t - \pi/3) = 2.2\cos(2\pi(5100)t + \pi/3) \]

\[ x_0(t) = 2.2\cos(2\pi(900)t - \pi/3) \]

\[ x_1(t) = 2.2\cos(2\pi(6900)t - \pi/3) \]

All map to \( x[n] \) when sampled at 6000 Hz. In other words these signals are aliases that a digital circuit cannot tell apart.
4.4) a) Graphical Method First Step: Find the spectrum of the information signal (the part with the lower frequency)

Then shift the spectrum to the positive and negative components of the carrier frequency and scale them by 1/2.

b) The signal is periodic since all the frequency components are multiples of 2000 Hz. The period is 1/2000Hz = 0.5 msec

f_s must be greater than twice the highest frequency component in the spectrum (12000 Hz). f_s must be greater than 24000 Hz.
4.8) a) To get the spectrum mathematically, find \( x(t) \) as a sum of complex exponentials:

\[
x(t) = \cos(50\pi t) \sin(700\pi t)
\]

\[
= \left[ \frac{1}{2} e^{j50\pi t} + \frac{1}{2} e^{-j50\pi t} \right] \left( \frac{1}{2} e^{j(700\pi t - \frac{\pi}{2})} + \frac{1}{2} e^{-j(700\pi t - \frac{\pi}{2})} \right)
\]

\[
= \frac{1}{4} e^{j50\pi t} e^{j(700\pi t - \frac{\pi}{2})} + \frac{1}{4} e^{j50\pi t} e^{-j(700\pi t - \frac{\pi}{2})} + \frac{1}{4} e^{-j50\pi t} e^{j(700\pi t - \frac{\pi}{2})} + \frac{1}{4} e^{-j50\pi t} e^{-j(700\pi t - \frac{\pi}{2})}
\]

\[
= \frac{1}{4} e^{j(750\pi t - \frac{\pi}{2})} + \frac{1}{4} e^{-j(750\pi t - \frac{\pi}{2})}
\]

\[
= \frac{1}{4} e^{\frac{j\pi T_s}{2}} e^{j750\pi t} + \frac{1}{4} e^{-\frac{j\pi T_s}{2}} e^{j650\pi t} + \frac{1}{4} e^{\frac{j\pi T_s}{2}} e^{-j650\pi t} + \frac{1}{4} e^{-\frac{j\pi T_s}{2}} e^{-j750\pi t}
\]

b) \( F_s \geq 2F_{\text{max}} \quad F_s \geq 750 \text{ Hz} \)

4.10) a)

\[
xx(\tau t) = \Re \left\{ e^{j(2\pi R \tau - \frac{\pi}{2})} \right\} = \cos \left( 2\pi 13\tau t - \frac{\pi}{2} \right)
\]

Notice that \( \tau t \) in the script is actually discrete (\( \tau t = nT_s \)). Since \( n \) is an integer:

\[
xx[n] = \cos \left( 2\pi 13n T_s - \frac{\pi}{2} \right) = \cos \left( 2\pi 13n T_s + 2\pi kn - \frac{\pi}{2} \right) = \cos \left( 2\pi (13T_s + k)n - \frac{\pi}{2} \right)
\]

Converting to continuous time:

\[
xx(t) = \cos \left( 2\pi \left( 13 + \frac{k}{T_s} \right) n T_s - \frac{\pi}{2} \right) = \cos \left( 2\pi \left( 13 + \frac{k}{T_s} \right) t - \frac{\pi}{2} \right)
\]

This means there is an alias for every integer value of \( k \). The plot function in Matlab connects consecutive points directly. This has the same effect of plotting the alias with the
smallest frequency. Choosing \( k = -1 \) gives the smallest frequency of 1.286 Hz (period of 0.7778 sec).

b) At least 20 samples per period is necessary to get a smooth plot. \( T_s = \frac{T_o}{20} = 0.0038 \)

4.11) a) This problem can be done fairly easily by inspection, if you understand how modulation affects the frequency spectrum. However, the problem asks specifically for phasors which can get a little tedious. On an exam, it is generally better to try the problem by inspection first. Then verify or complete your answer using phasors.

\[
\left[ 3 + \frac{1}{2} e^{j(\omega_1 t - \frac{\pi}{2})} + \frac{1}{2} e^{j(\omega_2 t + \frac{\pi}{2})} \right] \left[ \frac{1}{2} e^{j\omega_3 (1.5t + \frac{\pi}{2})} + \frac{1}{2} e^{j\omega_3 (-1.5t - \frac{\pi}{2})} \right] \\
\frac{1}{2} e^{j\omega_3 (1.5t + \frac{\pi}{2})} \left[ 3 + \frac{1}{2} e^{j(\omega_1 t - \frac{\pi}{2})} + \frac{1}{2} e^{j(\omega_2 t + \frac{\pi}{2})} \right] + \frac{1}{2} e^{j\omega_3 (-1.5t - \frac{\pi}{2})} \left[ 3 + \frac{1}{2} e^{j(\omega_1 t - \frac{\pi}{2})} + \frac{1}{2} e^{j(\omega_2 t + \frac{\pi}{2})} \right] \\
\frac{3}{2} e^{j\omega_3 (1.5t + \frac{\pi}{2})} + \frac{1}{4} e^{j(\omega_1 t + 1.5\omega_2 t + \frac{\pi}{2})} + \frac{1}{4} e^{j(-\omega_1 t + 1.5\omega_2 t + \frac{\pi}{2})} + \frac{3}{2} e^{j\omega_3 (-1.5t - \frac{\pi}{2})} + \frac{1}{4} e^{j(\omega_1 t - 1.5\omega_2 t - \frac{\pi}{2})} + \frac{1}{4} e^{j(-\omega_1 t - 1.5\omega_2 t - \frac{\pi}{2})} \\
+ \frac{1}{4} e^{j(\omega_1 t + 1.5\omega_2 t - \frac{\pi}{2})} + \frac{1}{4} e^{j(-\omega_1 t + 1.5\omega_2 t - \frac{\pi}{2})}
\]

\[
3 \cos \left( 13\pi t + \frac{\pi}{2} \right) + \frac{1}{2} \cos(14\pi t) + \frac{1}{2} \cos(12\pi t + \pi)
\]

\( \omega_1 = 12\pi, \quad \omega_2 = 13\pi, \quad \omega_3 = 14\pi \)

\( A_1 = 1/2, \quad A_2 = 3, \quad A_3 = \frac{1}{2} \)

\( \phi_1 = \pi, \quad \phi_2 = \pi/2, \quad \phi_3 = 0 \)
c) \( F_s \geq 2F_{\text{max}} \quad F_s \geq 14 \text{ Hz} \)
4.13)a) Unless otherwise specified, Discrete-to-Continuous converter will always reconstruct the components of \( y(t) \) with aliases that have frequencies less than \( f_s/2 \). If \( x(t) = y(t) \), all the frequency components of \( x(t) \) are less than \( f_s/2 \).

\[ f_s/2 \geq 150 \text{ Hz} \quad f_s \geq 300 \text{ Hz} \]

b) \[ x[n] = 2 \cos \left( 2\pi (50 + f_s k_1) n T_s + \frac{\pi}{2} \right) + \cos \left( 2\pi (150 + f_s k_2) n T_s \right) \]

Choose the values of \( k_1 \) and \( k_2 \) such that each cosine has the minimum frequency

\[ x[n] = 2 \cos \left( 2\pi (50 + 250 k_1) t/250 + \frac{\pi}{2} \right) + \cos \left( 2\pi (150 + 250 k_2) t/250 \right) \]

\[ = 2 \cos \left( 2\pi (50 + 250(0)) n/250 + \frac{\pi}{2} \right) + \cos \left( 2\pi (150 + 250(\pm 1)) n/250 \right) \]

\[ = 2 \cos \left( \frac{2\pi}{5} n + \frac{\pi}{2} \right) + \cos \left( -\frac{4\pi}{5} n \right) \]

\[ = 2 \cos \left( \frac{2\pi}{5} n + \frac{\pi}{2} \right) + \cos \left( \frac{4\pi}{5} n + \pi \right) \]

c) 

**Magnitude Spectrum of \( x[n] \)**

![Magnitude Spectrum of x[n]](image)

**Phase Angle Spectrum of \( x[n] \)**

![Phase Angle Spectrum of x[n]](image)
d)

\[ y(t) = 2 \cos \left( 2\pi (50)t + \frac{\pi}{2} \right) + \cos(2\pi(0)t) \]

\[ 2 \cos \left( 2\pi (50 + f_s k_1)t + \frac{\pi}{2} \right) + \cos(2\pi(150 + f_s k_2)t) \]

The first cosine has not changed from \( x(t) \): \( f_s \geq 100 \text{ Hz} \)

The second cosine has become a DC component: \( f_s < 300 \text{ Hz} \) and \( 150 + f_s k_2 = 0 \)

\[ f_s k_2 = -150 \]

\( k_2 \) is an integer. The only value that satisfies the constraints on \( f_s \) is \( k_2 = -1 \).

\[ f_s = 150 \text{ Hz} \]