HW7 Solutions ECE2000

Problems 3.1, 3.2, 3.5, 3.6, 3.8, 3.9, 3.12, 3.13, 3.19 (Note that 3.13 was not assigned and is not part of HW7)

Phasor: Phasors are a way of representing the Magnitude and Phase of a sinusoid as a complex constant. Using the inverse Euler’s formula, any sinusoid may be rewritten as 2 complex conjugate exponentials. The constant portion (in blue) of the exponential is the phasor for that exponential.

\[
A \cos(2\pi f_0 t + \varphi) = \frac{A}{2} e^{j(2\pi f_0 t + \varphi)} + \frac{A}{2} e^{-j(2\pi f_0 t + \varphi)} = \frac{A}{2} e^{j\varphi} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\varphi} e^{-j2\pi f_0 t}
\]

Note: There is a shortcut for adding sinusoids with the same frequency that does not require the negative complex conjugate or dividing the amplitude by 2.

\[
A_1 \cos(2\pi f_0 t + \varphi_1) + A_2 \cos(2\pi f_0 t + \varphi_2) = \Re\{A_1 e^{j\varphi_1} e^{j2\pi f_0 t}\} + \Re\{A_2 e^{j\varphi_2} e^{j2\pi f_0 t}\}
= \Re\{(A_1 e^{j\varphi_1} + A_2 e^{j\varphi_2}) e^{j2\pi f_0 t}\} = A \cos(2\pi f_0 t + \varphi)
\]

where \(Ae^{j\varphi} = A_1 e^{j\varphi_1} + A_2 e^{j\varphi_2}\)

Be sure to convert the phasors to rectangular form \((x+jy)\) to sum them. Again, you can only use this trick for summing sinusoids with the same frequency. Use the complex conjugate form for everything else including multiplying sinusoids (modulation) and plotting spectra.

Spectrum of a Sum of Sinusoids: We introduce the spectrum of a signal as a plot of the phasors of the complex exponentials that make up the signal vs. the frequency associated with each complex exponential. Spectra are often but not always presented with separate plots for the magnitude and phase portions of the phasors.

3.1) a)

\[10 \cos \left(800\pi t + \frac{\pi}{4}\right)\]

\[= \frac{10}{2} e^{j\pi/4} e^{j4000\pi t} + \frac{10}{2} e^{-j\pi/4} e^{-j2\pi4000t}\]

Notice that the magnitude of both exponentials is half the amplitude of the original cosine. These map to 2 vertical lines in the magnitude spectrum at the frequencies \((f_0\) and \(-f_0\)) of the corresponding complex exponentials. The phases map in a similar fashion into the phase angle spectrum. Notice that the phase of the exponential at the positive frequency is the
opposite of the phase of the one at the negative frequency.

The equation in problem 3.1a has a sum of 3 cosines of different frequencies with 2 complex exponentials each. This corresponds to 6 vertical lines in both the magnitude and phase angle spectrums. Be careful with negative sinusoids. Since amplitude is always positive, the sign is part of the phase. Negating a sinusoid is equivalent to adding π.

*Remember that sums of sinusoids of the same frequency can be combined into a single cosine.

b) A signal is periodic if it has a fundamental frequency ($F_0$). The fundamental frequency is the greatest common factor of the frequencies in the spectrum.

\[400 \text{ Hz} = (2 \times 2 \times 2 \times 2 \times 5 \times 5) \text{ Hz} \quad \text{2nd harmonic}^{*} \]
\[600 \text{ Hz} = (2 \times 2 \times 2 \times 3 \times 5 \times 5) \text{ Hz} \quad \text{3rd harmonic} \]
\[800 \text{ Hz} = (2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5) \text{ Hz} \quad 2 \times 2 = 4^{th} \text{ harmonic} \]
\[F_0 = (2 \times 2 \times 2 \times 5 \times 5) \text{ Hz} = 200 \text{ Hz} \]

$x(t)$ is periodic with a period of $T_0 = 1/F_0 = 1/(200 \text{ Hz}) = 5 \text{ msec}$

*Dividing out the fundamental frequency leaves the harmonic number.

c) Adding an additional cosine term to $x(t)$ adds 2 more vertical lines to the magnitude and phase spectrums

\[400 \text{ Hz} = (2 \times 2 \times 2 \times 2 \times 5 \times 5) \text{ Hz} \quad 4^{th} \text{ harmonic} \]
\[500 \text{ Hz} = (2 \times 2 \times 5 \times 5 \times 5) \text{ Hz} \quad 5^{th} \text{ harmonic} \]
\[600 \text{ Hz} = (2 \times 2 \times 2 \times 3 \times 5 \times 5) \text{ Hz} \quad 6^{th} \text{ harmonic} \]
\[800 \text{ Hz} = (2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5) \text{ Hz} \quad 8^{th} \text{ harmonic} \]

\[F = (2 \times 2 \times 5 \times 5) \text{ Hz} = 100 \text{ Hz} \]

$y(t)$ is also periodic with a period of

\[T_0 = 1/(100 \text{ Hz}) = 10 \text{ msec} \]
3.2) a) Simply find the complex exponential that corresponds to each vertical line and convert them to cosines.

\[ x(t) = 11e^{j(2\pi 0t)} + 7e^{-j\pi/3}e^{j2\pi(50t)} + 7e^{j\pi/3}e^{-j2\pi(50t)} + 4e^{-j\pi/2}e^{j2\pi(175t)} + 4e^{j\pi/2}e^{-j2\pi(175t)} \]

\[ = 11 + 14 \cos(2\pi 50t - \pi/3) + 8 \cos(2\pi 175t - \pi/2) \]

b) 50 Hz = (2 x 5 x 5) Hz  
175 Hz = (5 x 5 x 7) Hz  
\( f_0 = (5 x 5) \) Hz = 25 Hz  
\( T_0 = 1/(25 \text{ Hz}) = 40 \text{ msec} \)

Notice that the DC component of 11 does not have any effect on the fundamental frequency. Think of it as the 0th harmonic (0 x 5 x 5 Hz = 0 Hz).

c) Decomposing real periodic signals into complex exponentials produces complex conjugates of each frequency component, one of which always has negative frequency.

3.5)a) \( f_0 = \text{gcf}(0,100,250) = 50 \text{ Hz} \)

N is the harmonic number of the component with the largest frequency: 250/50 = 5

<table>
<thead>
<tr>
<th>( k^{th} ) Harmonic</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>-250</td>
<td>-200</td>
<td>-150</td>
<td>-100</td>
<td>-50</td>
<td>0</td>
<td>50</td>
<td>100</td>
<td>150</td>
<td>200</td>
<td>250</td>
</tr>
<tr>
<td>( a_k )</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>10e^{-j\pi/4}</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10e^{j\pi/4}</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

b) \( T_0 = 1/f_0 = 20 \text{ ms} \)  
\( x(t) \) is periodic

c)

3.6)a)

\[ (12 + 7 \sin(\pi t - \frac{\pi}{3})) \cos(13\pi t) = \left(12 + 7 \cos\left(\pi t - \frac{5\pi}{6}\right)\right) \cos(13\pi t) \]

\[ = \left(12 + \frac{7}{2}e^{-j\frac{5\pi}{6}}e^{j\pi t} + \frac{7}{2}e^{j\frac{5\pi}{6}}e^{-j\pi t}\right)\left(\frac{1}{2}e^{j\frac{13\pi}{6}} + \frac{1}{2}e^{-j\frac{13\pi}{6}}\right) \]
Multiply out the terms:

\[
\frac{1}{2}e^{j13\pi t} \left( 12 + \frac{7}{2} e^{-j5\pi/6} e^{jnt} + \frac{7}{2} e^{j5\pi/6} e^{-jnt} \right) + \frac{1}{2}e^{-j13\pi t} \left( 12 + \frac{7}{2} e^{-j5\pi/6} e^{jnt} + \frac{7}{2} e^{j5\pi/6} e^{-jnt} \right)
\]

\[
= \left( 6e^{j13\pi t} + \frac{7}{4} e^{-j5\pi/6} e^{j14\pi t} + \frac{7}{4} e^{j5\pi/6} e^{j12\pi t} \right) + \left( 6e^{-j13\pi t} + \frac{7}{4} e^{-j5\pi/6} e^{-j12\pi t} + \frac{7}{4} e^{j5\pi/6} e^{-j14\pi t} \right)
\]

Reorder terms into complex conjugate pairs:

\[
= \frac{7}{4} e^{j5\pi/6} e^{j12\pi t} + \frac{7}{4} e^{-j5\pi/6} e^{-j12\pi t} + 6e^{j13\pi t} + 6e^{-j13\pi t} + \frac{7}{4} e^{-j5\pi/6} e^{j14\pi t} + \frac{7}{4} e^{j5\pi/6} e^{-j14\pi t}
\]

Convert into a sum of cosines:

\[
= \frac{7}{2} \cos \left( 12\pi t + \frac{5\pi}{6} \right) + 6 \cos (13\pi t) + \frac{7}{2} \cos \left( 14\pi t - \frac{5\pi}{6} \right)
\]

3.8) \( \omega_0 = \text{gcf}(0, 40\pi, 60\pi, 120\pi) = 20\pi \text{ rad/cycle} \quad T_0 = \frac{2\pi}{\omega_0} = 10 \text{ ms} \)

\( N \) is the harmonic number of the component with the largest frequency: 60/10 = 6

<table>
<thead>
<tr>
<th>( k )th Harmonic</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-50</td>
<td>-40</td>
<td>-30</td>
<td>-20</td>
<td>-10</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>( a_k )</td>
<td>2e^{j\pi/3}</td>
<td>0</td>
<td>0</td>
<td>1.5e^{j\pi/2}</td>
<td>2e^{j\pi/5}</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2e^{j\pi/5}</td>
<td>1.5e^{j\pi/2}</td>
<td>0</td>
<td>0</td>
<td>2e^{j\pi/3}</td>
</tr>
</tbody>
</table>

b) The additional sinusoid adds 2 spectral lines at 25 Hz and -25 Hz with complex amplitudes of 5e^{j\pi/6} and 5e^{j\pi/6} respectively. The signal is still periodic with a fundamental frequency of 5 Hz and a fundamental period of 40 ms.
3.9)a)

b) Note that because the signal is periodic the window of integration can be shifted arbitrarily. Shifting the window around a single pulse simplifies the calculation.

\[ a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) dt \]

\[ = \frac{1}{T_0} \int_{-t_c}^{t_c} 1 dt = \frac{2t_c}{T_0} \]

c)

\[ a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 kt} dt = \frac{1}{T_0} \int_{-t_c}^{t_c} e^{-j\omega_0 kt} dt = \frac{e^{-j\omega_0 kt_c} - e^{j\omega_0 kt_c}}{-j\omega_0 kT_0} \]

\[ = \frac{1}{\omega_0 kT_0} \left[ e^{j(\omega_0 kt_c - \pi/2)} + e^{-j(\omega_0 kt_c + \pi/2)} \right] = \frac{2}{\omega_0 kT_0} \cos(\omega_0 kt_c - \pi/2) \]

d) Magnitude Spectrum with \( t_c = T_0/4 \)

e) Magnitude Spectrum with \( t_c = T_0/10 \)
f) The relative size refers to how large the amplitude of a component is relative to the other components. Decreasing $t_c$ increases the relative size of the high frequency components.

3.12)a) Time plot for $x(t)$

\[ a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) \, dt = \frac{1}{10} \int_0^{10} 2 \, dt = 1 \]

b) $a_1 = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 t} \, dt = \frac{1}{10} \int_0^{10} 2e^{-j\frac{2\pi}{10} t} \, dt = \frac{2}{10} e^{-j\frac{2\pi}{5}} - e^{-j\frac{\pi}{5}}$

\[ = \frac{e^{-j\frac{2\pi}{5}} - e^{-j\frac{\pi}{5}}}{-j\pi} = j \frac{1 - (-1)}{\pi} = j\frac{2}{\pi} \]

c) Adding a sinusoid only affects the spectrum at the frequency components associated with that sinusoid. Since constants are considered to be “sinusoids with zero frequency”, adding 1 to a signal increases the zero frequency component $a_0$ by 1. All other frequency components are unaffected.

3.13) $x(t) = \sin(10t) \sum_{k=-3}^{3} \frac{1}{1+jk} e^{jkt}$

\[ = \sum_{k=-3}^{3} \left[ \frac{1}{2j} \frac{1}{1+jk} e^{jkt} e^{j10t} \right] - \sum_{k=-3}^{3} \left[ \frac{1}{2j} \frac{1}{1+jk} e^{jkt} e^{-j10t} \right] \]

\[ = \sum_{k=-3}^{3} \left[ \frac{1}{2(j-k)} e^{j(k+10)t} \right] - \sum_{k=-3}^{3} \left[ \frac{1}{2(j-k)} e^{j(k-10)t} \right] \]

Converting the complex amplitude into a phasor can get tedious considering that there are 7 different values. Fortunately, MATLAB has no trouble calculating complex numbers.
% The summation represents a vector where only k differs for each value
k = -3:3; ak = abs(0.5./(j-k));
w = [-13:-7 7:13];

% Plot Magnitude Spectrum
stem(w, [ak ak], 'r', 'LineWidth', 2.5)
title('Magnitude Spectrum');
xlabel('Radial Frequency (rad/sec)');
ylabel('Amplitude');
xlim([min(w) max(w)]*1.1);
ylim([0 max(ak)*1.1]);
grid on;

Note that the positive and negative spectrums are centered around the carrier frequencies 10 and -10
3.19

a) Centered above x-axis ⇔ Contains positive DC component
   Single Sinusoid ⇔ One complex conjugate pair in spectrum
   Peak is slightly delayed ⇔ Phase of positive complex exponential is small and negative

Spectrum 3: \( f(t) = 2 + 3 \cos \left( 2\pi 1.2t - \frac{\pi}{4} \right) \)

b) Centered at x-axis ⇔ No DC component
   Single Sinusoid ⇔ One complex conjugate pair in spectrum

Spectrum 5: \( f(t) = 3 \cos(2\pi 1.5t + \pi) \)

c) Centered above x-axis ⇔ Contains positive DC component
   Single Sinusoid ⇔ One complex conjugate pair in spectrum
   t=0 is halfway between peak and trough ⇔ Phase of positive complex exponential is \( \frac{\pi}{2} \)

Spectrum 1: \( f(t) = 2 + 3 \cos \left( 2\pi 1.2t + \frac{\pi}{2} \right) \)
d) Sum of more than one Sinusoid $\Leftrightarrow$ Multiple complex conjugate pairs in spectrum

Pattern repeats slowly $\Leftrightarrow$ Complex exponentials have small frequencies

Spectrum 2: $f(t) = 3 \cos \left( 2\pi 0.6t - \frac{\pi}{4} \right) + 3 \cos(2\pi 1.5t + \pi)$

e) Sum of more than one Sinusoid $\Leftrightarrow$ Multiple complex conjugate pairs in spectrum

Pattern repeats more quickly than d) $\Leftrightarrow$ Complex exponentials have larger frequencies

Spectrum 4: $f(t) = 3 \cos \left( 2\pi 1.2t - \frac{\pi}{4} \right) + 3 \cos(2\pi 2t + \pi)$