Cut for Classical Core Logic

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Abstract

In an earlier paper in this JOURNAL I provided detailed motivation for the constructive and relevant system of *Core Logic*; explained its main features; and established that, even though Cut is not a rule *of* the system, nevertheless Cut is admissible for it, and indeed with potential epistemic gain. If Π is a proof of the sequent $\Delta : A$, and Σ is a proof of the sequent $\Gamma, A : \theta$, then there is a computable reduct $[\Pi, \Sigma]$ of some *subsequent* of the 'overall target sequent' $\Delta, \Gamma : \theta$. In the constructive case the potential epistemic gain consists in the possibility that the subsequent in question be a *proper* subsequent of $\Delta, \Gamma : \theta$, and indeed even *logically stronger* than $\Delta, \Gamma : \theta$.

In this paper it is established that Cut is likewise admissible for *Classical* Core Logic, which is obtained from Core Logic by adding a suitably relevantized form of the rule of Classical Dilemma. In the classical case there is an additional feature of potential epistemic gain: the proof $[\Pi, \Sigma]$ might have a *lower degree of non-constructivity* than do Π and Σ.

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1 Introduction

1.1 Core Logic and its classical extension

The system $\mathbb{C}$ of Core Logic, whose rules will be stated in §2, is a system of constructive and relevant deductive reasoning. It is methodologically adequate for all the needs of the *constructivist* reasoner. This claim can be justified by appeal to the following results, where $\vdash_I$ stands for deducibility in Intuitionistic Logic, and $\vdash_{\mathbb{C}}$ stands for deducibility in Core Logic (which is a *sub*system of Intuitionistic Logic):[1]

1. If $\Delta \vdash_I \bot$ then $\Delta \vdash_{\mathbb{C}} \bot$.
2. If $\vdash_I \varphi$ then $\vdash_{\mathbb{C}} \varphi$.
3. If $\Delta \vdash_I \varphi$ and $\Delta \nvdash_I \bot$, then $\Delta \vdash_{\mathbb{C}} \varphi$.
4. If $\Delta \vdash_I \varphi$, then either $\Delta \vdash_{\mathbb{C}} \varphi$ or $\Delta \vdash_{\mathbb{C}} \bot$.
5. $\nvdash_{\mathbb{C}} \bot$.

The system $\mathbb{C}^+$ of *Classical* Core Logic is the system of classical but relevant deductive reasoning that results from Core Logic by appending a suitably relevantized form of the rule of Classical Dilemma (for which, see §3.1). It is methodologically adequate for all the needs of the *classical* reasoner. This claim can likewise be justified by appeal to the following results, where $\vdash_C$ stands for deducibility in Classical Logic, and $\vdash_{\mathbb{C}^+}$ stands for deducibility in Classical Core Logic:[2]

1. If $\Delta \vdash_C \bot$ then $\Delta \vdash_{\mathbb{C}^+} \bot$.
2. If $\vdash_C \varphi$ then $\vdash_{\mathbb{C}^+} \varphi$.

[1] The following logical implications among these five statements are left as an exercise for the reader:

(4) implies (1).
(4) implies (3).
(4) and (5) jointly imply (2).
(1), (3) and (5) jointly imply (2).
(1) and (3) jointly imply (4), given Dilemma in the metalanguage on the claim $\Delta \vdash_I \bot$.

[2] See footnote 1; the same pattern of relationships holds among the five statements concerning the classical case. (Dilemma in the metalanguage in this case will be on the claim $\Delta \vdash_C \bot$.)

3. If $\Delta \vdash_C \varphi$ and $\Delta \nvdash_C \bot$, then $\Delta \vdash_{\mathbb{C}+} \varphi$.

4. If $\Delta \vdash_C \varphi$, then either $\Delta \vdash_{\mathbb{C}+} \varphi$ or $\Delta \vdash_{\mathbb{C}+} \bot$.

5. $\nvdash_{\mathbb{C}+} \bot$.

A diagram will be useful at the outset. It shows how Core Logic and its classicized counterpart, Classical Core, sit in relation to the well-known systems of Classical, Intuitionistic and Minimal Logic, which have well-behaved natural-deduction formulations; and in relation to the relevance logic R of Anderson and Belnap.[3] Note that Core Logic, indicated with dashed lines, is the intersection of Classical Core with Intuitionistic Logic.

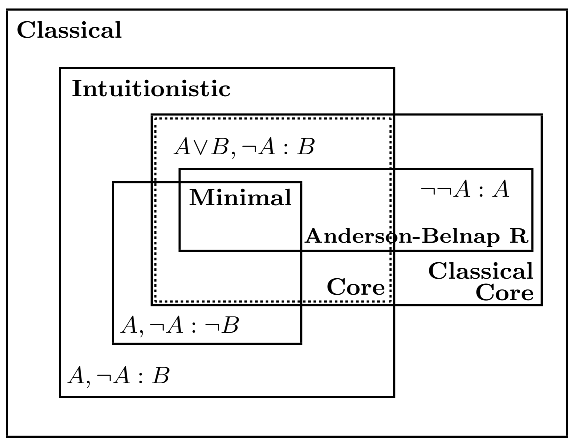

Figure 1: System Containments

Note that Core Logic contains Disjunctive Syllogism ($A \vee B, \neg A : B$), but even Classical Core Logic contains neither one of the two closely related Lewis paradoxes $A, \neg A : B$ and $A, \neg A : \neg B$.

Tennant [2012] set out a system of natural deduction for Core Logic, consisting of only introduction and elimination rules (suitably framed). The system has the signal features

(i) all elimination rules are in parallelized form, and

(ii) all major premises for eliminations stand proud.

[3] See Anderson and Belnap [1975].

By (ii), all proofs are in normal form. By (i) and (ii), natural deductions in Core Logic are essentially isomorphic to their corresponding sequent proofs, in a sequent calculus for Core Logic consisting of 'right-introduction' and 'left-introduction' rules that can straightforwardly be 'read off' the introduction- and elimination-rules, respectively, of the system of natural deduction for Core Logic. Sequent proofs in Core Logic are both cut-free and thinning-free. The only structural rule in the sequent calculus for Core Logic is the rule of initial sequents, $\varphi : \varphi$.

In Tennant [2012] the admissibility of Cut was proved for the system of Core Logic, in the following 'epistemically gainful' form.

Theorem 1 (Cut Elimination for Core Proof)
There is an effective method [,] *that transforms any two core proofs*

$$\begin{array}{ccl} \Delta & A, \Gamma & \\ \Pi & \Sigma & \textit{(where } A \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)} \\ A & \theta & \end{array}$$

into a core proof $[\Pi\ \Sigma]$ of θ or of $\perp$ *from (some subset of)* $\Delta \cup \Gamma$.

We choose the terminology of Cut Elimination, or the admissibility of Cut, even when the system in question is a natural deduction system rather than a sequent calculus, for the following reason. When a natural-deduction theorist speaks of normalizability, or of the composition of derivations, the underlying assumption is that the system *allows for the formation* of licit but *ab*normal proofs

$$\begin{array}{c} \Delta \\ \Pi \\ \underbrace{(A) \quad , \quad \Gamma} \\ \Sigma \\ \theta \end{array}$$

by means of proof-compositions (or proof-accumulation). These are supposedly effected by grafting (copies of) the proof Π of the conclusion A onto all occurrences of A as an undischarged assumption in another proof Σ. But from the point of view of the present study, such proof-accumulations do *not* themselves qualify as proofs, if any assumption-occurrence of A in Σ is the major premise of an elimination. From this point of view, in such cases the two proofs Π and Σ are not permitted to be combined into a single proof of the kind indicated by the foregoing display. One can, however, *operate on* parts of Π and parts of Σ in one's attempt to find a licit proof of (some subsequent of) $\Delta, \Gamma : \theta$. This computable operation is represented as the binary function $[\Pi\ \Sigma]$, which was defined in the course of proving Theorem 1.

The present study is a sequel to Tennant [2012]. It investigates *Classical* Core Logic, and can take over intact the explanation that was given in Tennant [2012] of the distinctive methods that are employed in order to ensure that proofs in the system maintain relevance of their premises to their conclusions. The present study aims to extend Theorem 1 to the system of *Classical* Core Logic, which, as already remarked, results from Core Logic by appending the rule of Classical Dilemma. Using Classical Dilemma rather than Classical Reductio has the advantage of potentially lowering the degree of non-constructivity of the reduct $[\Pi\,\Sigma]$. See §9 for further details.

1.2 The Core logician's method of relevantizing proofs

The distinction between classical and constructive methods of proof is by now so well known and understood that we need not be detained by any need to expound upon it. More useful for present purposes would be some words of explanation of the distinctive way in which the Core logician seeks to characterize the relation of *relevance* that, intuitively, the relevantist wishes to capture between the premises and the conclusion of any well-constructed proof.

The explication of relevance embodied in Core Logic (in its constructive or classical versions) differs from that of relevance logicians in the tradition of Anderson and Belnap [1975]. The best contrast can be found in the possible responses to Lewis's now famous little proof of the argument $A, \neg A : B$.[4] If one can prove the sequent

$$A : A \vee B$$

and can prove Disjunctive Syllogism:

$$A \vee B, \neg A : B,$$

then by a single application of Cut (construed as a *proof-forming* rule) one can prove

$$A, \neg A : B.$$

All relevantists wish to avoid proving this last result. Some relevantists go so far as to say that it is invalid. I do not. To be sure, it is semantically or

[4]I use the notation $\Delta : \varphi$ for sequents. These are statements of arguments: a set Δ of premises on the left, and a conclusion φ on the right. An alternative and frequently used notation is $\Delta \Rightarrow \varphi$.

model-theoretically valid in both the intuitionistic and the classical senses. But its premises lack that relation of relevance that we wish to see between premises and conclusions of arguments for which relevantists are to be able and willing to furnish proofs. The relevantist's reason for wishing to eschew proofs of the argument $A, \neg A : B$ is not that it is invalid, but that all proofs of it (must) fail to establish the relevance of its premises $A, \neg A$ to its conclusion B.

The relevantist's main concern is with how best to capture those inferences whose premises bear a relation of relevance to their conclusions (however that relation of relevance might be explicated). Here I am using 'relevance' in an intuitive, informal sense, in which there is no commitment to the sorts of views espoused by relevance logicians in the Anderson-Belnap tradition. So perhaps I should use a subscripted term, like 'relevance$_{ii}$', to indicate that. (Here, 'ii' is short for 'intuitive, informal'.)

Apart from the odd outlier in the tradition, virtually every relevantist has no objection to $\vee$-Introduction ($A : A \vee B$, and $B : A \vee B$). So, to avoid commitment by proof to Lewis's paradox ($A, \neg A : B$), there are only two possible choices:

1. Make it impossible, in one's system of relevance logic, to prove Disjunctive Syllogism ($A \vee B, \neg A : B$); or

2. Do not allow proof-*formation* by steps of (unrestricted) Cut.

The Anderson-Belnap tradition opted for (1), and hung on to unrestricted Cut as a *proof-forming* rule in their system R.

I opted rather for (2), and hung on to Disjunctive Syllogism, for the system of Core Logic. To summarize:

	A-B	NT
DS	No	Yes
Cut	Yes	No

Another point of contrast between the Anderson-Belnap tradition and (Classical) Core Logic is that, when imposing constraints in the interests of relevance$_{ii}$, the former focuses on the object-linguistic conditional arrow, whereas the latter focuses on the turnstile of deducibility, and is prepared to re-examine the orthodoxy that has ossified around the so-called structural rules.

1.3 An exigent sense in which core proofs maintain relevance of their premises to their conclusions

Core Logic, in both its constructive and its classical forms, is a *relevant* logic, in an interesting and deeper sense than that provided merely by the assurance that the logic does not allow derivation of the Lewis Paradox (in either its positive or its negative form). We shall confine ourselves here to the propositional system in explaining the formal explication of relevance$_{ii}$ that is to be had from Core Logic. This task involves spelling out the details of a very exigent form of 'variable-sharing' (or, in our terminology, sharing of *atoms*). To this end we need to supply the following definitions.

Definition 1 $\pm\varphi$ $\equiv_{df}$ *some atom occurs both positively and negatively in* φ. *(Note that* $\pm$ *is a metalinguistic predicate, not a function sign.)*

Definition 2 $\varphi \approx \Delta$ $\equiv_{df}$ *some atom has the **same** parity (positive or negative, at some occurrence) in* φ *as it has in some member of* Δ.

Definition 3 *Suppose* $\varphi \neq \psi$. *Then* $\varphi \bowtie \psi$ $\equiv_{df}$ *some atom has the **opposite** parity at some occurrence in* φ *from that which it has at some occurrence in* ψ.

Definition 4 $\varphi_1, \ldots, \varphi_n$ $(n > 1)$ *is a* $\bowtie$-path connecting φ_1 to φ_n in Δ $\equiv_{df}$ *for* $1 \leq i \leq n$, φ_i *is in* Δ, *and for* $1 \leq i < n$, $\varphi_i \bowtie \varphi_{i+1}$.

Definition 5 *A set* Δ *of formulae is* $\bowtie$-connected $\equiv_{df}$ *for all* φ, ψ *in* Δ, *if* $\varphi \neq \psi$, *then there is a* $\bowtie$*-path connecting* φ *to* ψ *in* Δ.

Definition 6 *A* component *of* Δ *is an inclusion-maximal* $\bowtie$*-connected subset of* Δ *(where the* $\bowtie$*-connections are established via members of* Δ*).*

A core proof of a conclusion φ from a set Δ of undischarged assumptions establishes that Δ is *relevantly connected both within itself and to* φ, in the sense that exactly one of the following three conditions holds:[5]

1. Δ is non-empty, φ is $\perp$, and:
   if Δ is a singleton $\{\delta\}$, then $\pm\delta$; otherwise, Δ is $\bowtie$-*connected.*

[5] This result was first proved in Tennant [1992], Chapter 9, for propositional Core Logic (there called *IR*, for 'intuitionistic relevant' logic). A fuller exposition of how to extend the result to both the first-order case and the classical case is planned for a later study.

2. Δ is non-empty, φ is not $\bot$, and:
   the components $\Delta_1, \ldots, \Delta_m$ $(m \geq 1)$ of Δ are such that for $1 \leq i \leq m$, we have $\varphi \approx \Delta_i$.

3. Δ is empty, φ is not $\bot$, and $\pm\,\varphi$.

Cases (1) and (3) cover the two logical extremes. In case (1) we have a proof of the joint inconsistency of the premises in Δ. In that case Δ itself is the only component of Δ. In case (2) we have a proof of a logical theorem φ. In that case φ will contain some atom both positively and negatively.

Case (2) covers the 'middle range', so to speak, and it is this case that reveals the most interesting structure involving both Δ and φ. First, the set Δ of premises is partitioned into components $\Delta_1, \ldots, \Delta_n$ $(n \geq 1)$, each of which, if not a singleton, is $\bowtie$-connected. Moreover, each component Δ_i bears a special relation to φ, to wit: some atom occurs with the same parity in φ as it does in *some member* of Δ_i.[6]

2 Rules for Core Logic

Core Logic lacks any proof-forming rule of Cut. Nevertheless, as already stressed, we have the very important result that Cut is *admissible* for Core Logic, in the sense given by Theorem 1 reported above.

In generalizing Theorem 1 here, so that it holds for *Classical* Core Logic, we of course (re-)prove Theorem 1 itself on the way. In fact, all that we really need to do in this paper is cover just those extra cases that are occasioned by the presence of the rule of Classical Dilemma, in the inductive step of the proof by induction that delivers the Cut-admissibility result. But we shall set out the proof of Cut-admissibility for Classical Core Logic in full detail, making use of a much more compact linear notation for proofs than the completely detailed two-dimensional schemata that were used in Tennant [2012]. So the present study is a sequel to Tennant [2012], subsuming it and making it much more succinct. Readers who wish either to have more of the motivational background for Core Logic, or to have more information about its relation to other well-studied systems, will find it in pp. 450–453 of Tennant [2012]. And in §2 of that paper they will find whatever extra structural detail they might wish to have filled in if the compact notation used here proves to be a little too terse.

[6] I am grateful to an anonymous referee for raising the point that the relevant notion of relevance merited some positive characterization within the context of this paper.

2.1 Notational conventions

The graphic statement below of the introduction and elimination rules that constitute Core Logic requires some preliminary explanation of notational conventions.

I follow here the notational convention of Prawitz [1965]: parenthetically enclosed numerals are used to label both the application of a discharge-rule, and the assumption-occurrences above it that are discharged by that application. With discharge rules, however, Core Logic requires attention to further features.

Boxes next to discharge strokes over 'assumptions for discharge' indicate that vacuous discharge is not allowed. That is to say, there really must be an assumption of the indicated form in the subordinate proof in question, available to be discharged upon application of the rule. With ($\wedge$-E) and ($\forall$-E) we require only that at least one of the indicated assumptions should have been used, and be available for discharge.

The *diamond* next to the discharge stroke in ($\rightarrow$-I(b)) below indicates that it is not required that the assumption φ in question should (have been used and) be available for discharge. But if it is available, then it is discharged.

The notation @ indicates places where the parameter a is not allowed to occur. This notation is used in the statement of the graphic version of the rules of $\exists$-Elimination and of $\forall$-Introduction, and the restrictions thereby effected are the familiar ones.

2.2 Graphic rules for Core Logic

The rules for Core Logic (the relevant and *constructive* system) as stated in Tennant [2012] can be stated graphically as follows. Note that each rule is *pure* in the sense that the only logical operator to occur in its statement is the logical operator that the rule in question concerns. The operator occurs only in dominant position, in the conclusion of the introduction rule and in the major premise of the corresponding elimination rule.

It is worth reminding the reader that all major premises for eliminations stand proud, with no proof-work above them. That is why, in the graphic form of the rules to follow, there are no occurrences of three descending dots above any of the major premises of E-rules.

(¬-I)

$$\dfrac{\begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \bot \end{array}}{\neg\varphi}\,(i)$$

(¬-E)

$$\dfrac{\neg\varphi \qquad \begin{array}{c} \vdots \\ \varphi \end{array}}{\bot}$$

(∧-I)

$$\dfrac{\begin{array}{c} \vdots \\ \varphi \end{array} \qquad \begin{array}{c} \vdots \\ \psi \end{array}}{\varphi \wedge \psi}$$

(∧-E)

$$\dfrac{\varphi \wedge \psi \qquad \begin{array}{c} \underbrace{[\varphi]^{(i)} , [\psi]^{(i)}} \\ \vdots \\ \theta \end{array}}{\theta}\,(i)$$

(∨-I)

$$\dfrac{\begin{array}{c} \vdots \\ \varphi \end{array}}{\varphi \vee \psi} \qquad \dfrac{\begin{array}{c} \vdots \\ \psi \end{array}}{\varphi \vee \psi}$$

(∨-E)

$$\dfrac{\varphi \vee \psi \quad \begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \theta \end{array} \quad \begin{array}{c} [\psi]^{(i)} \\ \vdots \\ \theta \end{array}}{\theta}\,(i) \qquad \dfrac{\varphi \vee \psi \quad \begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \bot \end{array} \quad \begin{array}{c} [\psi]^{(i)} \\ \vdots \\ \theta \end{array}}{\theta}\,(i) \qquad \dfrac{\varphi \vee \psi \quad \begin{array}{c} [\varphi]^{(i)} \\ \vdots \\ \theta \end{array} \quad \begin{array}{c} [\psi]^{(i)} \\ \vdots \\ \bot \end{array}}{\theta}\,(i)$$

(→-I(a))
$$\dfrac{\begin{array}{c}\square\text{---}(i)\\ \varphi \\ \vdots \\ \bot\end{array}}{\varphi\to\psi}\,(i)$$

(→-I(b))
$$\dfrac{\begin{array}{c}\Diamond\text{---}(i)\\ \varphi \\ \vdots \\ \psi\end{array}}{\varphi\to\psi}\,(i)$$

(→-E)
$$\dfrac{\varphi\to\psi \qquad \begin{array}{c} \\ \vdots \\ \varphi\end{array} \qquad \begin{array}{c}\square\text{---}(i)\\ \psi \\ \vdots \\ \theta\end{array}}{\theta}\,(i)$$

(∃-I)
$$\dfrac{\begin{array}{c}\vdots \\ \varphi^x_t\end{array}}{\exists x\varphi}$$

(∃-E)
$$\dfrac{\exists x\varphi \; @ \qquad \begin{array}{c}\square\text{---}(i)\\ \underbrace{@ \ldots \varphi^x_a \ldots @} \\ \vdots \\ \psi \; @\end{array}}{\psi}\,(i)$$

(∀-I)
$$\dfrac{\begin{array}{c}@ \\ \vdots \\ \varphi\end{array}}{\forall x\varphi^a_x}$$

(∀-E)
$$\dfrac{\forall x\varphi \qquad \begin{array}{c}(i)\text{---}\;\cdots\square\cdots\;\text{---}(i)\\ \underbrace{\varphi^x_{t_1}\,,\;\ldots\;,\;\varphi^x_{t_n}} \\ \vdots \\ \theta\end{array}}{\theta}\,(i)$$

2.3 Remarks on harmony.

Constructive or intuitionistic logicians in the recent philosophical tradition of logical reform pioneered by Dummett and Prawitz set great store by a special feature of pairs of introduction and elimination rules for the logical operators, which they call *harmony*. Intuitively or informally, the introduction and elimination rules @-I and @-E for a logical operator @ are in harmony with one another just in case by applying @-E to the conclusion of an application α of @-I, one cannot extract more than, nor can one be unable to extract as much as, already had to be established according to the conditions legitimating the application α of @-I.

This informal characterization has been explicated in at least three different ways by different authors. The original Gentzen–Prawitz explication (see Gentzen [1934, 1935] and Prawitz [1965]) appeals to the existence of a suitable *reduction procedure* for the operator @, in order to make formal and precise sense of the informal notion of harmony at issue between @-I and @-E. It is these reduction procedures, in effect, which collectively permit the proof of normalizability or Cut-admissibility for the whole proof system.

The second explication of harmony, due to Belnap (see Belnap [1962]), appeals to the way in which, upon extending a language L by adding @ as a new item to its logical vocabulary, the rules for operators already in L plus the new rules @-I and @-E result in a mere *conservative extension* of the deducibility relation in L. That is, if an L-sequent $\Delta : \varphi$ is derivable in the proof system for the extended language $L + @$, then it is derivable in the proof system for the unextended language L.

The third explication of harmony is my own, and was given in Tennant [1978]. It has undergone some modifications in response to various critiques, and the interested reader will find the most recent formulation of my proposed explication in Tennant [forthcoming]. The basic idea (for which the challenge is to provide suitably precise formulation) is that the conclusion of @-I is the strongest proposition that can feature as the conclusion of that rule (and this fact can be established by appealing to @-E, and making full use of it); while the major premise of @-E is the weakest proposition that can feature as the major premise of that rule (and this fact can be established by appealing to @-I, and making full use of it).

It would be gratifying if one could show that these three explications of the intuitive notion of harmony, when made suitably precise, are co-extensive (as has been done with all proposed formal explications of the notion of a computable function). But serious doubts have been raised about this prospect, most notably and forcefully by Steinberger (see Steinberger [2011]).

The present study rests content with the Gentzen–Prawitz explication of harmony in terms of reduction procedures, and indeed uses them in order to establish its main result. The new twist added is that reduction can, in general, be epistemically gainful. This, however, is a byproduct of the measures instituted to ensure the *relevance* of premises of core proofs (constructive or classical) to their conclusions. I do intend, however, to address in later work the subtle ways in which the *classical* logician might be able to argue that, in the 'core setting', it is the classical system rather than the constructive one that affords the better prospect of establishing the harmony of rule-pairs as explicated in the third way indicated above.[7]

3 Classicizing Core Logic

3.1 Rules of Classical Logic not derivable in Core Logic

The following strictly classical rules are interderivable in Core Logic. In each of these rules, it is the sentence φ that is its 'classical focus'. This is because the reasoner who applies the rule is presuming that φ is determinately true, or false, even though it might not be known which is the case.

The rule of *Classical Reductio ad Absurdum* (*CR*):

$$\dfrac{\begin{array}{c}\Box\overline{\quad\neg\varphi\quad}^{(i)}\\ \vdots \\ \bot\end{array}}{\varphi}(i)$$

The *Law of Excluded Middle* (*LEM*):

$$\dfrac{}{\varphi\vee\neg\varphi}$$

The rule of *Double Negation Elimination*:

$$\dfrac{\neg\neg\varphi}{\varphi}$$

[7] I am grateful to an anonymous referee for raising the point that the issue of harmony merited some discussion within the context of this paper.

The rule of *Classical Dilemma (Dil)* we state in two forms:

$$
\mathit{Dil}^{\varphi}_{\psi\psi}\;\frac{\begin{array}{cc} \overline{\varphi}^{\,(i)} & \overline{\neg\varphi}^{\,(i)} \\ \vdots & \vdots \\ \psi & \psi \end{array}}{\psi}\,(i)
\qquad
\mathit{Dil}^{\varphi}_{\psi\bot}\;\frac{\begin{array}{cc} \overline{\varphi}^{\,(i)} & \overline{\neg\varphi}^{\,(i)} \\ \vdots & \vdots \\ \psi & \bot \end{array}}{\psi}\,(i)
\quad , \quad \text{where } \psi \text{ is not } \bot.
$$
[8]

The notations $\mathit{Dil}^{\varphi}_{\psi\psi}$ and $\mathit{Dil}^{\varphi}_{\psi\bot}$ should be self-explanatory. I also use the labels '*Dil*' and 'Dil^{φ}' for Dilemma in either of these two forms.

Using any one of the foregoing strictly classical rules, one can derive any of the others, *modulo* Core Logic. One can therefore 'classicize' Core Logic by adjoining *any one* of these rules to the rules of Core Logic given in §2.2. I shall take Dilemma as my classical rule to obtain $\mathbb{C}^+$ from $\mathbb{C}$. Reasons for this choice will emerge below.

4 A compact notation for proofs

One achieves considerable brevity by exploiting a Polish notation for proofs, in which basic rules are represented as operators that form proofs from (sentences and) non-trivial immediate subproofs. An explanation in light of the rules $\wedge$-I and $\wedge$-E will suffice.

[8] The left subproof for an application of Dilemma is traditionally called the 'positive horn', and the right subproof the 'negative horn'. Note that whenever the positive horn of a would-be dilemma has $\bot$ as its conclusion, the effect of the apparently classical dilemma

$$
\frac{\begin{array}{cc} \overline{\varphi}^{\,(i)} & \overline{\neg\varphi}^{\,(i)} \\ \Pi_1 & \Pi_2 \\ \bot & \psi/\bot \end{array}}{\psi/\bot}\,(i)
$$

can be captured in Core Logic. Simply take

$$
\left[\begin{array}{cc} \overline{\varphi}^{\,(i)} & \\ \varphi & \neg\varphi \\ \Pi_1 & \Pi_2 \\ \dfrac{\bot}{\neg\varphi}\,(i) & \psi/\bot \end{array}\right].
$$

See Tennant [2012] for details about the 'cut-elimination' operation [,] on core proofs. Alternatively, the reader may simply continue with this paper. The positive-horn observation is owed to Jan von Plato (private communication).

1. The rule $\wedge$-I forms a proof of a conjunction from two non-trivial immediate subproofs, one for each conjunct. One can denote such a proof as
$$\wedge_I \Sigma_1 \Sigma_2 .$$
The reader knows, from the statement of the rule of $\wedge$-Introduction, that this proof has for its conclusion the conjunction of the conclusions of Σ_1 and Σ_2 respectively, and that its set of premises is the union of their respective sets of premises.

   One can also write
$$\wedge_I \Sigma_1 \Sigma_2 (\varphi \wedge \psi)$$
when it is important, in context, to know or to be reminded that the conclusion is $\varphi \wedge \psi$.

2. The rule $\wedge$-E forms a proof from a conjunction $\varphi \wedge \psi$, say, and a non-trivial immediate subproof Σ_1 among whose undischarged assumptions is at least one of the conjuncts φ, ψ. One can denote such a proof as
$$\wedge_E \Sigma_1 ,$$
in which mention is suppressed of the sentence $\varphi \wedge \psi$ featuring as the major premise for the terminal $\wedge$-E. The reader knows, from the statement of the rule of $\wedge$-Elimination, that this proof has as its conclusion that of its subproof Σ_1, and that its premises are those of Σ_1, minus φ, minus ψ, plus $\varphi \wedge \psi$.

   One can also write
$$\wedge_E \Sigma_1 \theta ,$$
when it is important, in context, to know or to be reminded that the conclusion is θ.

The notation
$$\exists_I \Xi(t)$$
will denote a proof of an existential formula $\exists x \varphi$, for some formula φ with just x free. The proof is formed by an application of $\exists$-I to the immediate subproof Ξ, whose conclusion is φ^x_t.

The notation
$$\forall_I \Xi(a)$$
will denote a proof of a universal formula $\forall x \varphi^a_x$, for some sentence φ involving a parameter a. The proof is formed by an application of $\forall$-I to

the immediate subproof Ξ, of conclusion φ, none of whose undischarged assumptions involves a.

The notation

$$\exists_E \Xi(a)$$

will denote a proof formed by an application of $\exists$-E to the immediate subproof Ξ. This step of $\exists$-E has major premise $\exists x\varphi$, for some formula φ with just x free. The sentence φ^x_a is an undischarged assumption of Ξ, and is the only one that involves the parameter a. Moreover, a does not occur in $\exists x\varphi$ or in the conclusion of Ξ. The reader knows, from the statement of the rule of $\exists$-Elimination, that this proof has as its conclusion that of its subproof Ξ, and that its premises are those of Ξ, minus φ^x_a, plus $\exists x\varphi$.

Finally, the notation

$$\forall_E \Xi(t_1, \ldots, t_n)$$

will denote a proof formed by an application of $\forall$-E to the immediate subproof Ξ. This step of $\forall$-E has major premise $\forall x\varphi$, for some formula φ with just x free. Moreover, its instantiations $\varphi^x_{t_1}, \ldots, \varphi^x_{t_n}$ are exactly the undischarged assumptions of Ξ of the form φ^x_t. The reader knows, from the statement of the rule of $\forall$-Elimination, that this proof has as its conclusion that of its subproof Ξ, and that its premises are those of Ξ, minus $\varphi^x_{t_1}$, …, minus $\varphi^x_{t_n}$, plus $\forall x\varphi$.

5 On two proofs connecting

I shall say that Π *connects with* Σ just in case the conclusion (call it A) of Π is a premise (i.e., an undischarged assumption) of Σ. So the two proofs may be rendered graphically as

$$\begin{array}{ccc} \Delta & & A, \Gamma \\ \Pi & \text{and} & \Sigma \\ A & & \theta \end{array} .$$

Here, A is called the *cut sentence*. When A is compound, we shall refer to its dominant operator as α. By virtue of A's being displayed separately in the premises of the proof Σ, it is to be assumed that $A \notin \Gamma$.

The *target sequent* is $\Delta, \Gamma : \theta$.

I shall on occasion also need to refer to the target sequent when we wish to use only the names of the two proofs involved, and know that the conclusion of one is a premise of the other. Suppose we are given a proof Π and a proof Σ, where the conclusion of Π is an undischarged assumption of Σ. Suppose we wish to accumulate Π and Σ. That is, we wish to 'graft' a copy of Π over each undischarged assumption-occurrence of A in Σ. I shall call the resulting target sequent $\Pi * \Sigma$. Note that $\Pi * \Sigma$ is a *sequent*, not the result of somehow combining the two proofs Π and Σ. In general, the reduct $[\Pi\ \Sigma]$ proves (some subsequent of) $\Pi * \Sigma$.

6 Cut-Elimination for Classical Core Logic

The central aim in this study is to extend the main result of Tennant [2012] for Core Logic to the system of *Classical* Core Logic that contains Dilemma as its classical rule. Classical Core Logic stands to Classical Logic as Core Logic stands to Intuitionistic Logic. Both the Core systems are relevantized: both are free of the two forms of the Lewis Paradox,

$$A, \neg A : B \qquad \text{and} \qquad A, \neg A : \neg B.$$

Theorem 2 (Cut Elimination for Classical Core Proof)
There is an effective method $[\ ,\ ]$ *that transforms any two proofs*

$$\begin{array}{lll} \Delta & A, \Gamma & \\ \Pi & \Sigma & \textit{(where } A \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)} \\ A & \theta & \end{array}$$

in Classical Core Logic into a proof $[\Pi\ \Sigma]$ *in Classical Core Logic* of θ or of $\perp$ *from (some subset of)* $\Delta \cup \Gamma$. *It may also be the case that* $[\Pi\ \Sigma]$ *is a proof in Core Logic, i.e., has no applications of the classical negation rule, even though at least one of* Π, Σ *does have such applications.*

Corollary 1 (Multiple Cut Elimination for Classical Core Proof)
One can effectively transform the core classical proofs

$$\begin{array}{llll} \Delta_1 & \Delta_n & A_1, \ldots, A_n, \Gamma & \\ \Pi_1, \ldots, & \Pi_n & \Sigma & \textit{(where } A_1, \ldots, A_n \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)} \\ A_1 & A_n & \theta & \end{array}$$

into a core classical proof $[\Pi_1\ \ldots\ [[\Pi_n\ \Sigma]\ldots]$ of θ or of $\perp$ *from (some subset of)* $\Delta_1 \cup \ldots \Delta_n \cup \Gamma$. *Moreover, it may also be the case that the latter proof is a proof in Core Logic, i.e., has no applications of the classical negation rule, even though at least one of* $\Pi_1, \ldots, \Pi_n, \Sigma$ *does have such applications.*

7 Proving Cut-Elimination for Classical Core Logic

This section is devoted to proving Theorem 2.

7.1 The complexity measure

The operation [,] is defined inductively on pairs $\langle \Pi, \Sigma \rangle$ of classical core proofs by means of a complexity measure on such pairs that takes into consideration (in descending order of importance):

1. the degree of non-constructivity of the proofs Π and Σ (as measured by the complexity of φ in applications of Dil^{φ});

2. the complexity of the cut-sentence A, when it occurs as the conclusion of either α-I or Classical Dilemma in Π, and as a major premise for α-E in Σ; and

3. these proofs' combined complexity (as measured by their numbers of steps).

The reader will easily verify that the transformations to be described below must, *after finitely many applications*, result in a proof of *some subsequent of the target-sequent* $\Pi * \Sigma$. The first emphasized feature flows from the fact that the operation [,] is made to apply, in the transforms, only to pairs $\langle \Pi', \Sigma' \rangle$ (of classical core proofs) that are less complex than $\langle \Pi, \Sigma \rangle$. We thereby effect a successful definition of the cut-elimination operation $[\Pi\ \Sigma]$ by recursion on the well-founded relation '$\langle \Pi', \Sigma' \rangle$ is less complex than $\langle \Pi, \Sigma \rangle$'. The second emphasized feature is established by inspection of each transformation.

7.2 Grounding cases and grounding conversions

First we take care of the *grounding cases*, by means of the following *grounding conversions*. These are the operations applied last in the cut-elimination or normalization process. Their results do not call for any further steps of cut-elimination or normalization. Note that the operation applies even when Π does *not* connect with Σ (see grounding cases (4) and (5) below)—that is, even if the conclusion of Π is not an undischarged assumption of Σ.

If Π *connects with* Σ:

1. $[A\ \Sigma] =_{df} \Sigma$. (Here, A is an undischarged assumption of Σ.)

2. $[\Pi\ A] =_{df}\ \Pi$. (Here, A is the conclusion of Π.)

3. If no assumption-occurrence of A within Σ is the major premise of an elimination, but A is an undischarged assumption of Σ, then

$$[\Pi\ \Sigma] =_{df} \begin{array}{c} \Delta \\ \Pi \\ (A)\ ,\ \Gamma \\ \Sigma \\ \theta \end{array}$$

If Π *does* not *connect with* Σ:

4. If the conclusion of $\Pi = \bot$, then $[\Pi\ \Sigma] =_{df}\ \Pi$.

5. If the conclusion of $\Pi \neq \bot$, then $[\Pi\ \Sigma] =_{df}\ \Sigma$.

In each grounding case the Π-transform of Σ establishes either $\bot$ or θ from some subset of:

$$(\text{premise-set of } \Pi) \cup (\text{premise-set of } \Sigma \setminus \{\text{conclusion of } \Pi\}).$$

Let us call any occurrence of A in Σ that is the major premise of an elimination an *MPE-occurrence of* A in Σ.

7.3 Ripe cases

It remains only to consider cases where Π and Σ satisfy the following conditions.

(i) Π is a non-trivial proof of A
(so we are not dealing with grounding case (1) above),

(ii) A is an undischarged assumption of Σ
(so we are not dealing with grounding cases (4) or (5) above), and

(iii) A has at least one MPE-occurrence in Σ
(so we are not dealing with grounding cases (2) or (3) above).

Cases satisfying conditions (i)-(iii) above will be called *ripe*.

A ripe case is one where the arguments Π, Σ for the operation $[\Pi, \Sigma]$ can be represented graphically by the annotated proof-schemata

$$\begin{array}{c}\\ \textit{non-trivial} \longrightarrow \\ \\ \end{array}\begin{array}{c}\Delta\\ \Pi\\ A\end{array} \qquad \begin{array}{c}\\ \textit{occurs as MPE} \nearrow \\ \\ \end{array}\begin{array}{c}A\,,\,\Gamma\\ \Sigma\\ \theta\end{array}$$

Ripe cases are of two kinds:

(a) *Soft ripe*: the last step of Σ does *not* have cut-sentence A as MPE;

(b) *Hard ripe*: the last step of Σ *does* have cut-sentence A as MPE.

Soft ripe cases are of three kinds, depending on the last step of Σ:

(1) *Soft ripe introductory*: the last step of Σ is an introduction;

(2) *Soft ripe eliminative*: the last step of Σ is an elimination;

(3) *Soft ripe dilemmatic*: the last step of Σ is one of Classical Dilemma.

Hard ripe cases are likewise of three kinds, depending on the last step of Π:

(i) *Hard ripe eliminative*: the last step of Π is a β-elimination, where β is not $\neg$;

(ii) *Hard ripe introductory*: the last step of Π is an α-Introduction;

(iii) *Hard ripe dilemmatic*: the last step of Π is one of Classical Dilemma.

We now have the following exhaustive and non-overlapping classification of cases. Remember that the dominant operator of the cut-sentence A (if A is compound) is being referred to as α. I indicate in square brackets the kind of transformation (to be described below) that will apply to the cases in question. We have already dealt with the Grounding conversions.

(I) Grounding cases [Grounding conversions]

(II) Ripe cases

- (a) Soft ripe cases
  - (1) Soft ripe introductory cases [I-Distribution conversions]
  - (2) Soft ripe eliminative cases [E-Distribution conversions]
  - (3) Soft ripe dilemmatic cases [*Dil*-Distribution conversions]
- (b) Hard ripe cases
  - (i) Last step of Π is β-E, $\beta \neq \neg$ [Permutation conversions]
  - (ii) Last step of Π is α-I [α-Reductions]
  - (iii) Last step of Π is *Dil* [*Dil*-Reductions]

7.4 Soft ripe cases: Distribution conversions

7.4.1 I-Distribution Conversions

In soft ripe introductory cases, the proof Σ ends with an introduction, which of course involves no MPE. In such a case we apply the appropriate I-Distribution conversion, depending on the dominant operator β in the conclusion of Σ. β-I involves either just one, or two, immediate subproofs.

When β-I involves just one immediate subproof Σ_1, we define

$$\begin{aligned}
\big[\,\Pi\ \beta_I\Sigma_1\,\big] \quad &=_{df}\ \ \big[\,\Pi\ \ \Sigma_1\,\big] \ \text{ if } \big[\,\Pi\ \ \Sigma_1\,\big] \text{ proves } \bot\text{; otherwise,}\\
&=_{df}\ \ \beta_I\big[\,\Pi\ \ \Sigma_1\,\big].
\end{aligned}$$

Let us abbreviate this definition by writing

$$\big[\,\Pi\ \beta_I\Sigma_1\,\big] \quad =_{df}\ \ \{\beta_I\}\big[\,\Pi\ \ \Sigma_1\big].$$

When β-I involves two immediate subproofs Σ_1 and Σ_2, we define

$$\begin{aligned}
\big[\,\Pi\ \beta_I\Sigma_1\Sigma_2\,\big] \quad &=_{df}\ \ \big[\,\Pi\ \ \Sigma_1\,\big] \ \text{ if } \big[\,\Pi\ \ \Sigma_1\,\big] \text{ proves } \bot\text{; otherwise,}\\
&=_{df}\ \ \big[\,\Pi\ \ \Sigma_2\,\big] \ \text{ if } \big[\,\Pi\ \ \Sigma_2\,\big] \text{ proves } \bot\text{; otherwise,}\\
&=_{df}\ \ \beta_I\big[\,\Pi\ \ \Sigma_1\,\big]\big[\,\Pi\ \ \Sigma_2\,\big]\ \ .
\end{aligned}$$

Let us abbreviate this definition by writing

$$\big[\,\Pi\ \beta_I\Sigma_1\Sigma_2\,\big] \quad =_{df}\ \ \{\beta_I\}\big[\,\Pi\ \ \Sigma_1\,\big]\big[\,\Pi\ \ \Sigma_2\,\big]\ \ .$$

7.4.2 E-Distribution Conversions

In soft ripe eliminative cases, the proof Σ ends with an elimination β-E, whose MPE is not the cut-sentence A. In such a case we apply the appropriate E-Distribution conversion. β-E involves either just one, or two, immediate subproofs.

When β-E involves just one immediate subproof Σ_1 (i.e. for $\beta = \neg, \wedge, \exists, \forall$) we define

$$\begin{aligned}\big[\Pi\ \ \beta_E\Sigma_1\big] \quad &=_{df} \quad \big[\Pi\ \ \Sigma_1\big] \text{ if } \big[\Pi\ \ \Sigma_1\big] \text{ proves a subsequent of } \Pi * \beta_E\Sigma_1; \\ &\qquad\quad \text{otherwise,} \\ &=_{df} \quad \beta_E\big[\Pi\ \ \Sigma_1\big]\end{aligned}$$

Let us abbreviate this definition by writing

$$\beta\text{-E Distribution} \qquad [\Pi\ \ \beta_E\Sigma_1] \ = \ \{\beta_E\}[\Pi\ \ \Sigma_1]$$

When β-E involves two immediate subproofs Σ_1 and Σ_2, we define

$$\begin{aligned}\big[\Pi\ \ \beta_E\Sigma_1\Sigma_2\big] \quad &=_{df} \quad \big[\Pi\ \ \Sigma_1\big] \text{ if } \big[\Pi\ \ \Sigma_1\big] \text{ proves a subsequent of } \Pi * \beta_E\Sigma_1\Sigma_2; \\ &\qquad\quad \text{otherwise,} \\ &=_{df} \quad \big[\Pi\ \ \Sigma_2\big] \text{ if } \big[\Pi\ \ \Sigma_2\big] \text{ proves a subsequent of } \Pi * \beta_E\Sigma_1\Sigma_2; \\ &\qquad\quad \text{otherwise,} \\ &=_{df} \quad \beta_E\big[\Pi\ \ \Sigma_1\big]\big[\Pi\ \ \Sigma_2\big]\end{aligned}$$

Let us abbreviate this definition by writing

$$\beta\text{-E Distribution} \qquad [\Pi\ \beta_E\Sigma_1\Sigma_2] \ = \ \{\beta_E\}[\Pi\ \Sigma_1\Sigma_2]$$

The I- and E-Distribution conversions all enjoy the common form

$$[\Pi\ \rho\Sigma_1(\Sigma_2)] \ = \ \{\rho\}[\Pi\ \Sigma_1]\,([\Pi\ \Sigma_2]).$$

7.4.3 *Dil*-Distribution Conversions

Dil-Distribution Conversions deal with soft ripe cases where the proof Σ ends with an application of Classical Dilemma (i.e., where $\rho = Dil$).

$$[\Pi \;\; Dil\, \Sigma_1 \Sigma_2] \;=_{df}\; \{Dil\}[\Pi\ \Sigma_1]\,[\Pi\ \Sigma_2]$$

This is the form already encountered above, for E-Distribution Conversions when β-E involves two immediate subproofs. The placeholder ρ is replaced, in this instance, by *Dil*.

7.5 Hard ripe cases

Recall the classification above of sub-cases of hard ripe cases:

(i)	Last step of Π is β-E, $\beta \neq \neg$	[Permutation conversions]
(ii)	Last step of Π is α-I	[α-Reductions]
(iii)	Last step of Π is *Dil*	[*Dil*-Reductions]

7.5.1 Hard ripe eliminative cases: Permutation conversions

The Permutation conversions are as follows.[9]

$$\begin{aligned}
[\beta_E \Pi_1 \;\; \Sigma] \;&=_{df}\; \{\beta_E\}[\Pi_1\ \Sigma] \qquad \text{if } \beta \text{ is } \wedge, \exists \text{ or } \forall;\\
[\rightarrow_E \Pi_1 \Pi_2 \;\; \Sigma] \;&=_{df}\; \{\rightarrow_E\}\Pi_1[\Pi_2\ \Sigma];\\
[\vee_E \Pi_1 \Pi_2 \;\; \Sigma] \;&=_{df}\; \{\vee_E\}[\Pi_1\ \Sigma][\Pi_2\ \Sigma].
\end{aligned}$$

Every sequence of Distribution and Permutation conversions is clearly at most finitely long.

7.5.2 Hard ripe introductory cases: Reductions

Assume that A occurs as the MPE of the terminal step in Σ. Then the following transformations, called Reductions, apply. They are stated by revealing a certain level of structure within each of the proofs Π and Σ. So the reader must bear in mind that references to Π on the right-hand sides

[9] The three lines that follow summarize the Permutation conversions whose fully detailed statement occupied pp. 464–476 in Tennant [2012].

are to the first argument of the [,]-operation on the left-hand side.[10]

$$\begin{array}{lll}
[\neg_I \Pi_1 \;\; \neg_E \Sigma_1] & =_{df} & [[\Pi \;\; \Sigma_1]\, \Pi_1] \\
[\wedge_I \Pi_1 \Pi_2 \;\; \wedge_E \Sigma_1] & =_{df} & [\Pi_1\, [\Pi_2 \;\; [\Pi \;\; \Sigma_1]]] \\
[\vee_I \Pi_i \;\; \vee_E \Sigma_1 \Sigma_2] & =_{df} & [\Pi_i \;\; [\Pi \;\; \Sigma_i]] \quad (i = 1, 2) \\
[\rightarrow_{I(a)} \Pi_1 \;\; \rightarrow_E \Sigma_1 \Sigma_2] & =_{df} & [[\Pi \;\; \Sigma_1] \;\; \Pi_1] \\
[\rightarrow_{I(b)} \Pi_1 \;\; \rightarrow_E \Sigma_1 \Sigma_2] & =_{df} & [[\Pi \;\; \Sigma_1]\, [\Pi_1 \;\; [\Pi \;\; \Sigma_2]]] \\
[\exists_I \Pi_1(t) \;\; \exists_E \Sigma_1(a)] & =_{df} & [\Pi_1 \;\; [\Pi \;\; {\Sigma_1}^a_t]] \\
[\forall_I \Pi_1(a) \;\; \forall_E \Sigma_1(t_1, \ldots, t_n)] & =_{df} & [{\Pi_1}^a_{t_1} \ldots [{\Pi_1}^a_{t_n} \;\; [\Pi \;\; \Sigma_1]] \ldots]
\end{array}$$

Note that the embedded terms of the form $[\Pi \;\; \Sigma_i]$ serve to take care, recursively, of other possible assumption-occurrences, within Σ, of the cut sentence A.

The proof of cut elimination for Core Logic, as in Tennant [2012], is now complete. This is why we said, in §1.2, that in generalizing Theorem 1 so that it holds for *Classical* Core Logic, we of course (re-)prove Theorem 1 itself on the way. In order to extend the proof to cover the classical case, we need only consider now how to handle applications of the rule of Dilemma.

7.6 Hard ripe dilemmatic cases: *Dil*-Reductions

Assume that the cut sentence is the conclusion of an application of Classical Dilemma at the terminal step in Π, and occurs as the MPE of the terminal step in Σ.

First we need to define the following operation.

$$\begin{array}{llll}
\langle Dil^{\varphi} \rangle \Xi\Theta & =_{df} & \Xi & \text{if } \Xi \text{ does not use } \varphi\text{; otherwise} \\
 & =_{df} & \Theta & \text{if } \Theta \text{ does not use } \neg\varphi\text{; otherwise} \\
 & =_{df} & [\neg_I \Xi \neg\varphi \;\; \Theta] & \text{if } \Xi \text{ proves } \bot\text{; otherwise} \\
 & =_{df} & Dil^{\varphi}\Xi\Theta. &
\end{array}$$

We can then perform a *Dil*-Reduction in accordance with the following definition:

$$[Dil^{\varphi}\, \Pi_1 \Pi_2 \;\; \Sigma] =_{df} \langle Dil^{\varphi} \rangle [\Pi_1 \;\; \Sigma]\, [\Pi_2 \;\; \Sigma].$$

[10]The seven lines that follow summarize the Reductions whose fully detailed statement occupied pp. 477–478 in Tennant [2012].

Note that in the case where both Π_1 and Π_2 prove the cut sentence A, the terms of the form $[\Pi_i \;\; \Sigma]$ on the right-hand side already take care of all potential assumption-occurrences of A in Σ. Thus there is no need for any embeddings of Π-transforms on the right-hand side.

Theorem 2 has now been proved, for the system of Classical Core Logic that is obtained by adopting Classical Dilemma as the sole classical rule.

8 What if one used *CR* instead of *Dil*?

I implement here the leading idea of von Plato and Siders [2012], who used *CR* as their classical negation rule. The *CR*-Distribution Conversion is straightforward:

$$[\Pi \;\; \mathit{CR}\,\Sigma_1] \;=_{df}\; \mathit{CR}\,[\Pi \;\; \Sigma_1]$$

But with *CR*-Reductions, the matter is more complicated. The classical focus of *CR* in Π is A, the cut sentence, and this can change to the possibly more complex classical focus θ of *CR* in the reduct.

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(\neg A_1) \;\; \neg_E \Sigma_1 \bot] \\ =_{df} \; & [\neg_I \neg_E [\Pi\,\Sigma_1] \neg\neg A_1 \;\; \Pi_1] \bot \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(A_1 \vee A_2) \;\;\; \vee_E \, \Sigma_1 \Sigma_2 \bot] \\ =_{df} \; & [\neg_I (\vee_E [\Pi\,\Sigma_1][\Pi\,\Sigma_2] \bot) \neg(A_1 \vee A_2) \;\; \Pi_1] \bot \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(A_1 \vee A_2) \;\;\; \vee_E \, \Sigma_1 \Sigma_2 \theta] \\ =_{df} \; & \mathit{CR}\,[\neg_I (\neg_E (\vee_E [\Pi\,\Sigma_1][\Pi\,\Sigma_2] \theta) \bot) \neg(A_1 \vee A_2) \;\; \Pi_1] \theta \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(A_1 \rightarrow A_2) \;\;\; \rightarrow_E \Sigma_1 \Sigma_2 \theta] \\ =_{df} \; & \mathit{CR}\,[\neg_I (\neg_E (\rightarrow_E [\Pi\,\Sigma_1][\Pi\,\Sigma_2] \theta) \bot) \neg(A_1 \rightarrow A_2) \;\; \Pi_1] \theta \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(A_1 \wedge A_2) \;\; \exists_E \Sigma_1 \theta] \\ =_{df} \; & \mathit{CR}\,[\neg_I (\neg_E (\wedge_E [\Pi\,\Sigma_1] \theta) \bot) \neg(A_1 \wedge A_2) \;\; \Pi_1] \theta \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(\exists x A) \;\; \exists_E \Sigma_1 \theta] \\ =_{df} \; & \mathit{CR}\,[\neg_I (\neg_E (\exists_E [\Pi\,\Sigma_1] \theta) \bot) \neg(\exists x A) \;\; \Pi_1] \theta \end{aligned}$$

$$\begin{aligned} & [\mathit{CR}\,\Pi_1(\forall x A) \;\; \forall_E \Sigma_1 \theta] \\ =_{df} \; & \mathit{CR}\,[\neg_I (\neg_E (\forall_E [\Pi\,\Sigma_1] \theta) \bot) \neg(\forall x A) \;\; \Pi_1] \theta \end{aligned}$$

For $\alpha = \wedge$, $\exists$ or $\forall$, whose elimination rules involve only one immediate subproof, we have the common pattern

$$
\begin{aligned}
&[CR\,\Pi_1(A)\ \ \alpha_E\Sigma_1\theta]\\
=_{df}\ & CR\,[\neg_I(\neg_E(\alpha_E[\Pi\,\Sigma_1]\theta)\bot)\neg A\ \ \Pi_1]\theta
\end{aligned}
$$

For $\alpha = \vee$ or $\rightarrow$, whose elimination rules involve two immediate subproofs, we have the common pattern

$$
\begin{aligned}
&[CR\,\Pi_1(A_1\alpha A_2)\ \ \alpha_E\Sigma_1\Sigma_2\theta]\\
=_{df}\ & CR\,[\neg_I(\neg_E(\alpha_E[\Pi\,\Sigma_1][\Pi\,\Sigma_2]\theta)\bot)\neg(A_1\alpha A_2)\ \ \Pi_1]\theta
\end{aligned}
$$

9 Degrees of Non-Constructivity: *Dil* v. *CR*

I submit that it is methodologically ill-advised to try to make do with Classical Reductio as one's sole classical negation-rule. The reason for this is interesting, and merits comment.

It is the classical foci φ in the classical negation-rules (*not* the sentence ψ in the rule of Classical Dilemma) that are being 'treated classically' when the rule in question is applied. When the classical reasoner applies a classical negation rule, she is taking the classical focus to be *determinately truth-valued*, independently of the means one might have for deciding what its truth-value is.

Now, the *degree of non-constructivity* of a strictly classical proof is measured by the highest degree of its classical foci. If the classical foci in a proof are all quantifier-free, that proof enjoys the lowest degree of non-constructivity. The next such degree up is when the most complex classical focus is Σ^0_1. This is the highest degree of non-constructivity permitted in proofs by users of *Markov's Rule*—essentially, Classical Reductio where φ can be at most Σ^0_1.

The rules of Classical Reductio and Double Negation Elimination go hand-in-hand, as do the Law of Excluded Middle and the rule of Classical

Dilemma. Witness:

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{}{\neg\varphi}\,(i)\\ \vdots\\ \bot\end{array}}{\neg\neg\varphi}\,(i)}{\varphi}\,{}_{DNE} \qquad {}_{CR}\cfrac{\cfrac{\neg\neg\varphi \qquad \cfrac{}{\neg\varphi}\,(i)}{\bot}}{\varphi}\,(i) \qquad \Bigg| \qquad {}_{Dil}\cfrac{\cfrac{(i)\cfrac{}{\varphi}}{\varphi\vee\neg\varphi} \qquad \cfrac{\cfrac{}{\neg\varphi}\,(i)}{\varphi\vee\neg\varphi}}{\varphi\vee\neg\varphi}\,(i) \qquad \cfrac{{}_{LEM}\cfrac{}{\varphi\vee\neg\varphi} \quad \begin{array}{c}\cfrac{}{\varphi}\,(i)\\ \vdots\\ \psi\end{array} \quad \begin{array}{c}\cfrac{}{\neg\varphi}\,(i)\\ \vdots\\ \psi\end{array}}{\psi}\,(i)$$

We see that

$$DNE, \neg\text{-I} \ \vdash CR;$$

$$CR, \neg\text{-E} \ \vdash DNE;$$

$$Dil, \vee\text{-I} \ \vdash LEM;$$

$$LEM, \vee\text{-E} \ \vdash Dil.$$

I shall take as the representatives from each pair Classical Reductio and Classical Dilemma, respectively. Note that Classical Dilemma, as stated above, has φ as its classical focus. This heightens the interest of the following observation.

Observation 1 *There is a derivation of Classical Dilemma:*

$$\cfrac{\begin{array}{c}\Box\cfrac{}{\varphi}\,(i)\\ \Pi_1\\ \psi\end{array}\qquad \begin{array}{c}\Box\cfrac{}{\neg\varphi}\,(i)\\ \Pi_2\\ \psi/\bot\end{array}}{\psi}\,(i)$$

using Classical Dilemma as its only classical rule of negation, in which ψ is the classical focus.

Proof.

$$\cfrac{(j)\cfrac{}{\psi} \qquad \begin{array}{c}\cfrac{\cfrac{(j)\cfrac{}{\neg\psi}\qquad \begin{array}{c}\cfrac{}{\varphi}\,(i)\\ \Pi_1\\ \psi\end{array}}{\bot}}{\neg\varphi}\,(i)\\ \Pi_2\\ \psi/\bot\end{array}}{\psi}\,(j)$$

Note that Dilemma is applied only at the terminal step, marked (j); and at that application its classical focus is ψ, not φ. Note also that the *left horn* for the terminal Dilemma, namely

$$\begin{array}{c}\dfrac{\dfrac{\neg\psi \qquad \begin{array}{c}\overline{}^{\,(i)}\\ \varphi\\ \Pi_1\\ \psi\end{array}}{\bot}}{\neg\varphi}{\scriptstyle(i)}\\ \Pi_2\\ \psi/\bot\end{array}\ ,$$

might not be in normal form, because the conclusion $\neg\varphi$ at the step (i) of $\neg$-I therein might stand as the major premise for a step of $\neg$-E in Π_2. So the latter proof-schema ought rather to be denoted as

$$\left[\begin{array}{cc}\dfrac{\dfrac{\neg\psi \qquad \begin{array}{c}\overline{}^{\,(i)}\\ \varphi\\ \Pi_1\\ \psi\end{array}}{\bot}}{\neg\varphi}{\scriptstyle(i)} & \begin{array}{c}\neg\varphi\\ \Pi_2\\ \psi/\bot\end{array}\end{array}\right].$$

Similar remarks will apply at all places subsequently, when proofs (in normal form) are 'accumulated' in such a way as to produce a result that might not be in normal form.

The upshot of Observation 1 is that when one has proofs

$$\begin{array}{c}\Delta,\varphi\\ \Pi_1\\ \psi\end{array} \qquad \text{and} \qquad \begin{array}{c}\Gamma,\neg\varphi\\ \Pi_2\\ \psi\end{array}\ ,$$

one can use Classical Dilemma to construct a proof of ψ from Δ, Γ in such a way as to ensure that, whenever φ and ψ are of different complexities, the classical focus is on the *less complex* of the two.

Observation 2 *Any derivation of Classical Dilemma using Classical Reductio as the sole classical negation rule ends with an application of Classical Reductio.*

Proof. The only rules which can be used in such a derivation are ¬-I, ¬-E and *CR*. Of these, only the latter can have the needed general form ψ for the conclusion.

The upshot of Observation 2 is that when one has proofs

$$\begin{array}{c}\Delta,\varphi\\ \Pi_1\\ \psi\end{array} \quad\text{and}\quad \begin{array}{c}\Gamma,\neg\varphi\\ \Pi_2\\ \psi\end{array},$$

where ψ is *more complex* than φ, one *cannot* use Classical Reductio to construct a proof of ψ from Δ, Γ in such a way as to ensure that the classical focus is on (the less complex) φ.

The best one can do, with Classical Reductio as one's only classical rule of negation, is reason as follows:

$$\dfrac{\dfrac{{\scriptstyle (j)}\overline{\neg\psi} \qquad \begin{array}{c}\Gamma, \dfrac{\dfrac{{\scriptstyle (j)}\overline{\neg\psi} \qquad \begin{array}{c}\Delta, \overline{\varphi}^{\,(i)}\\ \Pi_1\\ \psi\end{array}}{\bot}}{\neg\varphi}{\scriptstyle (i)}\\ \Pi_2\\ \psi\end{array}}{\bot}}{\psi}{\scriptstyle CR\,(j)} .$$

(If Π_2 has conclusion $\bot$, then the bottommost step of ¬-E is not needed.)

This, however, makes ψ the classical focus, because it is so for the terminal application of Classical Reductio. (The latter step is the only strictly classical step taken.) Within this derivation, the sentence φ is no longer being treated classically.

Note by contrast that the rule of Classical Reductio:

$$\begin{array}{c}\Box\overline{\neg\varphi}^{\,(i)}\\ \Pi\\ \dfrac{\bot}{\varphi}{\scriptstyle (i)}\end{array},$$

in which it is the sentence φ that is the classical focus, can be derived by appeal to Classical Dilemma as follows:

$$
\textit{Dil.}\;\dfrac{\begin{array}{c}{}^{(i)}\overline{}\\ \varphi\end{array} \qquad \begin{array}{c}\overline{}^{\,(i)}\\ \neg\varphi\\ \Pi \\ \bot\end{array}}{\varphi}\,(i)
$$

Within this derivation it is still φ that features as the sentence that is being treated classically. The classical focus has *not* shifted. The phenomenon of shifting classical focus manifests itself only when one goes from using Classical Dilemma to using Classical Reductio in its stead.

The case for preferring Classical Dilemma to Classical Reductio (as the true revealer of *loci* of non-constructivity within classical proofs) is now complete.

I conclude that it is advisable to seek to classicize Core Logic by adopting *only* Classical Dilemma as one's strictly classical negation-rule. For it is Classical Dilemma that affords the most stable focus on those sentence-occurrences that are being treated classically (non-constructively); hence, that gives us a firmer grasp on the degree of non-constructivity of any strictly classical proof. In §8 we pinpointed the way in which the choice of Classical Reductio (as proposed by von Plato and Siders [2012]) leads to shiftings of classical focus in their normalization process, and is therefore to be avoided.

It is worth noting for Classical Core based on Dilemma that the method of 'cut elimination' affords a *new* kind of potential epistemic gain: the happy circumstance of finding that one can constructivize certain proofs that might (mistakenly) be thought to be irremediably non-constructive. For, whenever the positive-horn subproof for a purported application of Classical Dilemma turns out to reduce to a *dis*proof (as can happen when we apply our cut-elimination methods), the net effect of the would-be Dilemma, as pointed out in foonote 8, is *constructively* obtainable.

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