N. TENNANT

A PROOF-THEORETIC APPROACH TO ENTAILMENT

0. INTRODUCTION

This paper is the technical counterpart, with a proof of the conjecture of my paper ‘Entailment and Proofs’ (Proc. of the Aristotelian Society, 1979). There I gave the philosophical motivation for a proof-theoretic approach to the problem of entailment. Here I preserve the spirit of that approach, with minor modifications of earlier definitions. It is hoped that the results of this paper will go some way towards a general theory of entailment and proof structure, a crucially underdeveloped aspect of extant theories of entailment, which is only gestured towards in Anderson and Belnap’s encyclopaedic coverage (pp. 216–217). My own conviction is that entailment is irreducibly proof-based, and I see no need for further semantical underpinnings than those disclosed in the final Corollaries below.

Before plunging into the proof theoretic workings below, the reader may benefit from some details of informal motivation. My basic aim is to avoid the Lewis paradoxes: $A, \sim \vdash B, A \vdash B \lor \sim B$, while retaining as much (that is, as many of the proofs) of classical logic as possible. In particular, disjunctive syllogism: $A \lor B, \sim A \vdash B$ is to be retained; also, as far as possible, the transitivity of proof so vital to the development of mathematics. My basic method is to attend to “good” and “bad” features of proof in natural deduction in order to isolate some notion of Proof as a satisfactory explication of the notion of entailment. Crudely put, the Lewis paradoxes must not be Provable; but there must be enough Proofs to do all of first order mathematics.

So consider the following four proofs of the first Lewis paradox in the ($\sim, \&$)-fragment of classical natural deduction:

\[
(1) \quad \frac{A \quad \sim A}{\sim B}
\]


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(2) \[ \frac{A}{\Lambda} \quad (1) \]
\[ \frac{\sim B}{A \& \sim B} \]
\[ \frac{A}{\Lambda} \quad (1) \]
\[ \frac{\sim A}{A \& \sim B} \]
\[ \frac{A}{\sim (A \& \sim B)} \quad (1) \]
\[ \frac{\Lambda}{(A \& \sim B)} \quad (2) \]

(3) \[ \frac{A \& \sim B}{\sim B} \quad (2) \]
\[ \frac{A \& \sim B}{\sim A} \]
\[ \frac{\Lambda}{A \& \sim B} \quad (1) \]
\[ \frac{A}{\sim A} \]
\[ \frac{\Lambda}{(A \& \sim B)} \quad (2) \]
\[ \frac{\Lambda}{B} \quad (3) \]

Proof (1) shows that we must ban the absurdity rule, or "ex falso quodlibet". Proofs (2) and (3) show that "spurious" assumptions for discharge can be smuggled in via maximal occurrences of formulae. These occurrences are maximal in Prawitz's sense: they stand as conclusions of introductions and as major premises of the corresponding eliminations. Finally, proof (4) shows that "spurious" assumptions for discharge can be smuggled in via yet another kind of maximal formula occurrence: maximal this time in the sense that they stand as conclusions of classical reductio and as major premises of eliminations.

Our sought notion of proof, then, must at the very least exclude these maximal formula occurrences and quodlibets. It turns out, indeed, that this is just enough: defining a Proof to be a proof in normal form with no quodlibets, we find that the Lewis paradoxes are not Provable, and yet
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Proofs suffice for all mathematics. This sufficiency is evident in the results that every inconsistent set of sentences is Provably so, and every consequence of a consistent set is Provable therefrom. Moreover, the thorny problem of transitivity is nicely solved in a way whose details will emerge below.

The main focus of attention is the connection between ordinary proofs and Proofs. This falls into two halves. A normalization theorem tells us how to get normal proofs from proofs, and an "extraction" theorem (Theorem 2) tells us how to get Proofs from normal proofs.

The normalization theorem is an essential advance on that of Prawitz for classical natural deduction. It is proved here, as his was not, for the full system with \( \vee \) and \( \exists \) primitive. But it is not a simple generalization of his theorem. For, in converting a proof into normal form Prawitz first applies transformations that ensure that all conclusions of reductio are \textit{atomic}. Thus obviously no conclusions of reductio in his normal proofs can stand as major premises of eliminations. Now I also secure this latter feature for my normal proofs, without, however, resorting to the drastic means of "atomicizing" all conclusions of reductio. My transformation procedure does away with conclusions of reductio standing as major premises of eliminations, but in general leaves complex conclusions of reductio otherwise untouched. Thus for me, every substitution instance of a normal proof is normal; while for Prawitz this is not so. Moreover, I still have the Subformula Theorem for normal proofs, which is what really matters.

If the reader is really only interested in its applications to entailment, he might skip the rather long proof of the normalization theorem. It would be considerably simplified by disregarding \( \vee \) and \( \exists \). But precisely because their inclusion is the main advance on Prawitz, I have worked the proof in full with attendant complications. Rather than leave the reader the difficult task of generalizing a method so as to include \( \vee \) and \( \exists \), I leave the simpler task of narrowing it so as to exclude them.

1. PRELIMINARIES

In his monograph 'Natural Deduction' Prawitz remarks on the "disturbing effect" of the classical reductio rule
(R) \[ \frac{\sim A (i)}{A} \]

in the system of natural deduction with \( \lor \) and \( \exists \) primitive. He proves a normalization theorem for a system without \( \lor \) and \( \exists \). In this paper we prove a normalization theorem for the full classical system with \( \sim, \&, \lor, \supset, \exists \) and \( \forall \) primitive, by suitably redefining the notion of normal form.

In our system \( \Lambda \) will be a constant occurring in proofs only to record the occurrence of a contradiction. The introduction and elimination rules (I rules and E rules) are as in my 'Natural Logic'. Applications of \( R \) or \( \sim I \) that do not discharge assumptions of the forms indicated:

\[ \frac{\sim A (i)}{A (i)} \quad \frac{A (i)}{\Lambda (i)} \]

will be regarded as applications of the 'absurdity rule'

(A) \[ \frac{\Lambda}{A} \]

Proofs are trees of sentence occurrences built up using these rules in the familiar way.

A \( \varphi \)-constellation in a proof \( \Pi \) is a locally maximal subtree of occurrences of \( \varphi \) in \( \Pi \), consisting, in the case of more than one occurrence, of minor premisses of \( \lor E \) and/or \( \exists E \). We use the notation

\[ \varphi \ldots \varphi \]

\[ \varphi \]

\[ \varphi \]

to suggest this. The bottommost occurrence of \( \varphi \) is said to be the vertex of the constellation.

A proof is in normal form if and only if

(i) every \( \varphi \)-constellation whose vertex is the minor premiss of an application of \( \sim E \) whose major premiss \( \sim \varphi \) does not
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A proof-theoretic approach to entailment stands as a conclusion has topmost occurrences of \( \varphi \) standing as conclusions only of introductions or eliminations (other than \( \forall \varphi \) and \( \exists \varphi \)).

(ii) every \( \varphi \)-constellation whose vertex is the major premiss of an elimination (is an MPE) has topmost occurrences of \( \varphi \) standing as conclusions only of eliminations (other than \( \forall \varphi \) and \( \exists \varphi \)).

Constellations violating these conditions will be called unwanted, and are of two distinct kinds, according as they are counterexamples to (i) or (ii) respectively:

(I) \[
\begin{array}{c}
\varphi \\
\vdots \\
\varphi \\
\hline
\varphi \\
\end{array}
\]
where at least one topmost occurrence of \( \varphi \) is a conclusion of \( R \) or \( A \)

(II) \[
\begin{array}{c}
\varphi \\
\vdots \\
\varphi \\
\hline
\varphi \\
\end{array}
\]
where at least one topmost occurrence of \( \varphi \) is a conclusion of \( R, A \) or an \( I \) rule

Henceforth we assume that all constellations mentioned are unwanted.

The degree of a \( \varphi \)-constellation is the degree of \( \varphi \). To find the companion of a \( \varphi \)-constellation \( \nabla \) proceed as follows: pass down from the vertex via MPE's as far as possible. The first sentence occurrence thus encountered that is not an MPE is the companion of \( \nabla \). For a constellation of type (I) the companion is just the vertex. For one of type (ii), the general picture is

(\[
\begin{array}{c}
\varphi \\
\vdots \\
\varphi \\
\hline
\varphi \\
\nabla \\
\nabla \\
\nabla \\
\hline
\varphi \\
\varphi_1 \\
\vdots \\
\varphi_n \end{array}
\]

(\[
\begin{array}{c}
\varphi_0 \Sigma_1^0 \Sigma_2^0 \\
\vdots \\
\varphi_{n-1} \Sigma_1^{n-1} \Sigma_2^{n-1} \\
\varphi_n \end{array}
\]

where \( n > 0, \varphi_1, \ldots, \varphi_{n-1} \) are MPE's, but the companion \( \varphi_n \) is not. The \( \Sigma_i \) are the subproofs of the minor premisses (if any) of the elimination with major premiss \( \varphi_i \) and conclusion \( \varphi_{i+1} \).
Different constellations may have the same companion. Given a companion, however, there is exactly one route upwards to vertices of constellations it accompanies. If non-trivial this route is via MPE's. The first constellation of highest degree among those to be encountered on this upward route is said to be illuminated by the companion. Every constellation has a unique companion, which illuminates a unique (but possibly different) constellation.

2. STRUCTURAL LEMMATA

LEMMA 1. Suppose the \( \varphi \)-constellation \( \nabla \) is connected by the downward route \( \varphi, \ldots, \varphi_{n-1}, \varphi_n \) from its vertex \( \varphi \) to its companion \( \varphi_n \) \((n > 0)\). Suppose the \( \Psi \)-constellation \( \nabla' \) is connected by the downward route \( \Psi, \ldots, \Psi_m, \varphi_n \) from its vertex \( \Psi \) to \( \varphi_n \). Then the companion of \( \nabla' \) is one of \( \Psi, \ldots, \Psi_m, \varphi_n \).

Proof. From the vertex \( \Psi \) of \( \nabla' \) there is only one route downwards, namely that followed to \( \varphi_n \), and it is this route that must be followed in search of the companion of \( \nabla' \). It cannot pass beyond \( \varphi_n \) on the way to the companion of \( \nabla' \), since \( \varphi_n \), being the companion of \( \nabla \), is not an MPE. The result follows.

We say that \( \nabla \) is higher than \( \nabla' \) in a proof just in case their companions \( \varphi \) and \( \varphi' \) are distinct and there is a route downwards from \( \varphi \) to \( \varphi' \).

LEMMA 2. In any abnormal proof \( \Theta \) we can find some \( \nabla \) with respect to which the proof \( \Theta \) has one of the following two forms:

\[
\begin{align*}
\text{(I)} & \quad (i) \sim \varphi \\
\Pi_j & \quad \Pi_k \\
\Lambda & \quad \Pi' \\
(\varphi(A)) \quad (i) \varphi(R) \quad \Pi_m \varphi(I \text{ or } E, \text{ but not } \forall E \text{ or } \exists E) \\
\nabla & \sim \varphi \\
\Lambda & \\
\Sigma
\end{align*}
\]
where $\Pi_j, \Pi_k, \Pi_m$ represent all possible forms of proofs of topmost occurrences of $\varphi$, and at least one of the forms $\Pi_j, \Pi_k, \Pi_m$ must feature;

(i) $\triangledown$ is illuminated by its own vertex, and is of maximum degree in $\Theta$; and

(ii) all constellations in the proofs of topmost occurrences of $\varphi$ are of lower degree than $\triangledown$.

(II)

(i) $\neg \varphi$

\[
\Pi_j \quad \Pi_k
\]

(i) $\varphi(R) \quad \varphi(A) \quad \Pi_l (I) \quad \Pi_m (E, \text{but not } \forall E \text{ or } \exists E)$

\[
\varphi \quad (\Sigma_1^0)(\Sigma_2^0)
\]

\[
\varphi_1 \quad \vdots
\]

\[
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1})
\]

\[
\varphi_n
\]

$\Sigma$

where $\Pi_j, \Pi_k, \Pi_l, \Pi_m$ represent all possible forms of proofs of topmost occurrences of $\varphi$, and at least one of the forms $\Pi_j, \Pi_k, \Pi_l, \Pi_m$ must feature;

(i) $\triangledown$ is illuminated by $\varphi_n (n > 0)$, and is of maximum degree in $\Theta$;

(ii) in order to pass upwards from $\varphi_n$ to some other $\triangledown'$ of equal degree one would have to pass through the vertex of $\triangledown$ and thereafter not through any of the topmost occurrences of $\varphi$ in $\triangledown$, but instead through one of the major premises for $\forall E$ or $\exists E$ in $\Theta$ that help to create $\triangledown$; and

(iii) $\varphi, \ldots, \varphi_{n-1}$ are MPE's, but $\varphi_n$ is not.

Proof. We can find an illuminated constellation $\triangledown$ of maximum degree than which none such is higher. $\triangledown$ is of type (I) or type (II).
CASE I. $\nabla$ is illuminated. Thus any constellation $\nabla'$ of maximum degree (i.e. that of $\nabla$), which occurs in a proof of one of the topmost occurrences of $\varphi$ in $\nabla$ would, by Lemma 1 and the fact that all occurrences of $\varphi$ in $\nabla$ above the vertex are minor premises of $\forall \varphi$ or $\exists \varphi$, stand higher than $\nabla$, contrary to choice of $\nabla$. Thus conditions (i) and (ii) are satisfied.

CASE II. Similarly. Any upward route from $\varphi_n$ violating condition (ii) would have to pass through a minor premiss of an elimination, thereby ensuring that the companion of any constellation $\nabla'$ of maximum degree that can be so reached stands above $\varphi_n$ (by Lemma 1) and therefore that $\nabla'$ is higher than $\nabla$, contrary to choice of $\nabla$.

3. Transformations

We now consider a general pattern of transformation:

$$
\frac{\varphi (\Sigma_1)(\Sigma_2)}{\Psi} \\
[\Pi, \overline{\Sigma}, \varphi]
$$

with the following instances:

$$
\frac{\Lambda}{\sim A} \\
\frac{A}{A} \\
\frac{\Lambda}{\Lambda}
$$
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\[ \frac{A}{\forall x A^q} \rightarrow \Pi'(t) \]

\[ \frac{A^q}{\Pi'}(A) \]

\[ \frac{B \Sigma}{\Pi'} \]

\[ \frac{A \rightarrow B \ A}{B} \]

\[ \frac{\Pi' \overline{A_1} \overline{A_2} \Pi'} \]

\[ \frac{A_i \Sigma_1 \Sigma_2}{B} \rightarrow (A_i) \]

\[ \frac{A_1 \lor A_2 \ B \ B}{B} \]

\[ \frac{\Pi' \overline{A_2^x} \ Pi'} \]

\[ \frac{A_i^x \Sigma}{\overline{\exists x A} B \ B} \rightarrow A_i^x \]

\[ \frac{\Sigma(t)}{B} \]

We have now prepared the ground for the description of a procedure whereby an abnormal proof \( \Theta \) may be transformed into a proof \( \Theta' \) of the same conclusion from a subset of the original assumptions, which, moreover, is less abnormal than \( \Theta \) in the following sense:

the maximum degree of constellations in \( \Theta' \) is no greater than in \( \Theta \); and, if the same, then \( \Theta' \) has fewer constellations of this degree than \( \Theta \).

4. THE PROCEDURE

Let \( \Theta \) be abnormal. By Lemma 2 \( \Theta \) has form (I) or (II).

In Case (I) \( \Theta' \) will be
Note that since $\Pi_m$ ends with an $I$ rule or an $E$ rule other than $\forall E$ or $\exists E$, its conclusion $\varphi$ cannot be the vertex of a 'new' $\varphi$-constellation (of type I). Thus by $I(i) \Theta'$ is as required.

In Case (II) we may need several steps of transformation before reaching a $\Theta'$ as required. We describe these steps in turn.

First we transform $\Theta$ to

$$\Theta_1 : \begin{array}{c}
\varphi_1 \\
\vdots \\
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1}) \\
\varphi_n \\
\end{array} \rightarrow \begin{array}{c}
(1) \\
(2) \\
\end{array}$$

$$\begin{array}{c}
\frac{\Pi_j}{\Lambda(1)} \\
\frac{\Pi_k}{\Lambda(2)} \\
\end{array} \begin{array}{c}
\varphi_1 \\
\vdots \\
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1}) \\
\varphi_n \\
\end{array} \rightarrow \begin{array}{c}
\Pi_m \\
\end{array} \begin{array}{c}
\varphi (\Sigma_1^0)(\Sigma_2^0) \\
\varphi_1 \\
\vdots \\
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1}) \\
\varphi_n \\
\end{array}$$

We now check for the possible occurrences of 'new' constellations in $\Theta_1$ of equal or higher degree than $\nabla$.

1. If the conclusion $\varphi_1$ of $[\Pi_1, \Sigma^0, \varphi]$ were now the vertex of such a constellation, then in $\Theta \varphi_1$ would have been a conclusion of $\forall E$ or $\exists E$, and would have been the vertex of a constellation of equal or higher degree than $\nabla$, contrary to condition (i) that $\varphi_n$ illuminates $\nabla$. 

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2. 'New' copies in $\Theta_1$ of constellations in $\Theta$ above $\varphi_n$ are of lower degree than $\nabla$, by condition (ii).

3. If the minor premiss $\varphi_n$ of $\sim E$ at top left in $\Theta_1$ is the vertex of a constellation (of type I) of equal or higher degree than $\nabla$, we perform the transformation for Case (I).

4. If the new $\varphi_n$-constellation is unwanted then, since the vertex $\varphi_n$, being $\nabla$'s companion in $\Theta$, is not an MPE in $\Sigma$, the constellation is of type I, and we can perform the transformation for Case (I).

5. If our transform is still not as required, this can only be because some conclusion $\sim \varphi$ of $\sim I$ at top left in $\Theta_1$ is (the vertex of) a constellation. In order to focus on such an occurrence of $\sim \varphi$, consider the relevant fragment $\Theta_2$ of $\Theta_1$:

$$
\Theta_2:
\begin{array}{c}
\varphi_n \\
\Sigma_1^n \\
\vdots \\
\Sigma_2^n \\
\varphi_{n-1} \\
\Sigma_1^{n-1} \\
\vdots \\
\Sigma_2^{n-1} \\
\varphi \\
\Sigma_1 \\
\vdots \\
\Sigma_2 \\
\Lambda \\
\sim \varphi
\end{array}
$$

where, without loss of generality, we are assuming that no transformation under (3) above was required. With respect to this occurrence of $\sim \varphi$, $\Pi_j$ has the form

$$
\begin{array}{c}
\varphi \\
\sim \varphi \\
\Lambda \\
\Pi_j \\
\Lambda
\end{array}
$$
and so $\Theta_2$ has the form

$$
\begin{array}{c}
(1) \\
\varphi \quad (\Sigma_1^0)(\Sigma_2^0) \\
\varphi_1 \\
\vdots \\
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1}) \\
\varphi_n \\
\hline
\Pi_j^1 \\
\Lambda \\
\varphi \\
\hline
\sim \varphi \\
(1) \\
\hline
\Lambda \\
\Pi_j^2 \\
\hline
\varphi_n \\
\sim \varphi \\
(2)
\end{array}
$$

Transform this to

$$
\Theta_3 : \\
\Pi_j^1 \\
\varphi \quad (\Sigma_1^0)(\Sigma_2^0) \\
\varphi_1 \\
\vdots \\
\varphi_{n-1}(\Sigma_1^{n-1})(\Sigma_2^{n-1}) \\
\varphi_n \\
\hline
\Lambda \\
\Pi_j^2 \\
\hline
\varphi_n \\
\sim \varphi \\
(2)
$$

Suppose that in $\Theta_3$ the conclusion $\varphi$ of $\Pi_j^1$ is the vertex of a 'new' constellation $\mathcal{N}$. If any topmost occurrence of $\varphi$ in $\mathcal{N}$ is a conclusion of $R$ or $A$, then $\Pi_j$ contains a $\varphi$-constellation (of type I), contrary to $\Pi(ii)$. Thus all topmost occurrences of $\varphi$ in $\mathcal{N}$ are conclusion of $I$ rules or $E$ rules. Thus we can apply to $\Theta_3$ the transformation for Case (II). The result will be of the form.
where considerations parallel to (1)-(4) above apply. Moreover (5) does not apply, since topmost occurrences of \( \varphi \) in \( \forall' \) are not conclusions of \( R \).

Suppose in \( \Theta_4 \) \( \varphi_n \) is the vertex of a constellation (of type I) of equal or higher degree than \( \forall \). Then we perform the transformation for Case (I). The resulting transform of \( \Theta_2 \) is now substituted for \( \Theta_2 \) in \( \Theta_1 \). The result is \( \Theta' \) as required.

We now have the

\[
\Delta
\]

NORMALIZATION THEOREM. Every classical proof \( \Theta \) can be transformed

\[
\Gamma
\]

into a proof \( \Theta' \) (\( \Gamma \subseteq \Delta \)) in normal form.

\[
\varphi
\]

Proof. By finitely many iterations of the procedure just described.

Obviously every proof in normal form can be pruned further so as to ensure that all applications of \( \forall E \) and \( \exists E \) discharge assumptions in the subordinate proofs as required. Henceforth we assume that this condition is satisfied by proofs in normal form.

5. APPLICATIONS TO ENTAILMENT

DEFINITIONS. A Proof is a proof in normal form containing no applications of the absurdity rule. Corresponding notions of Provability, Deducibility, etc., are defined in the usual way.
[\Pi] is the set of all sentences occurring in \Pi.
|\Delta| is the set of all subformulae of members of \Delta.
\overrightarrow{\Delta} is the set of all negations of members of \Delta.
\overline{\Pi} is the set of assumptions discharged by R in \Pi.

(Note that \overrightarrow{A} is a subformula of both \forall xA and \exists xA; and for simplicity we regard \Lambda as a subformula of \sim A. We write \varphi < \Psi for \varphi is a subformula of \Psi.

\Delta

LEMMA 3. Suppose \Pi is in normal form and the constellation \varpi of which
φ
the conclusion \varphi is the vertex has topmost occurrences of \varphi standing as conclusions only of \sim E, \&E, \lor E or \forall E. Then \varphi \in |\Delta|.

Proof. For the purposes of considerations developed below, consider
\Delta
\Pi with this much detail, to be uncovered as we go along:
φ

\phi

\overrightarrow{\Psi_0(\alpha_1)}
\overrightarrow{\Psi_1(\alpha_2)}
\overrightarrow{\Psi_2(\alpha_3)}
\overrightarrow{\Psi_n(\alpha_n)}
\overrightarrow{\Varpi}

\overrightarrow{\Gamma_0}
\overrightarrow{\Gamma_1}
\overrightarrow{\Gamma_n}
\overrightarrow{\Sigma_0}
\overrightarrow{\Sigma_1}
\overrightarrow{\Sigma_n}

\overrightarrow{\theta_0(\alpha_0)}
\overrightarrow{\theta_1}
\overrightarrow{\theta_2}
\overrightarrow{\theta_n}

If \varphi \in \Delta we are done.
If \varphi \notin \Delta choose some topmost occurrence of \varphi in \varpi.
If it is discharged, it will be so by some application \alpha of \forall E or \exists E in
\varpi, with major premiss \theta_1; and we have \varphi < \theta_1.
If it is not discharged, then by the condition on \( \lor \) it is the conclusion of some application (\( \alpha_0 \)) of \( \sim E, \& E, \lor E \) or \( \forall E \) with major premiss \( \theta_0 \); so we have \( \varphi < \theta_0 \). Let \( \theta_0 \) stand as the conclusion of \( \Sigma_0 \) with undischarged assumptions \( \Gamma_0 \). Since \( \Pi \) is in normal form, \( \Sigma_0 \) satisfies the hypothesis of the Lemma. Thus by IH \( \theta_0 \in |\Gamma_0| \). Say \( \theta_0 < \Psi_0 \in \Gamma_0 \).

If \( \Psi_0 \notin \Delta \) we are done.

If \( \Psi_0 \notin \Delta \) then \( \Psi_0 \) is discharged by some application (\( \alpha_1 \)) of \( \forall E \) or \( \exists E \) in \( \Delta \), with major premiss \( \theta_1 \); and we have \( \varphi < \theta_0 < \Psi_0 < \theta_1 \).

Thus if \( \varphi \notin \Delta \) we have \( \varphi < \theta_1 \) where \( \theta_1 \) is as described.

If \( \theta_1 \notin \Delta \) we are done.

If \( \theta_1 \notin \Delta \) consider the proof \( \Sigma_1 \), with undischarged assumptions \( \Gamma_1 \), of which it is the conclusion. Since \( \Pi \) is in normal form, \( \Sigma_1 \) satisfies the hypothesis of the Lemma. Thus by IH \( \theta_1 \in |\Gamma_1| \). Say \( \theta_1 < \Psi_1 \in \Gamma_1 \).

If \( \Psi_1 \notin \Delta \) we are done.

If \( \Psi_1 \notin \Delta \) then \( \Psi_1 \) is discharged by some application (\( \alpha_2 \)) below (\( \alpha_1 \)) of \( \forall E \) or \( \exists E \) in \( \Delta \), with major premiss \( \theta_2 \); and we have \( \varphi < \theta_1 < \Psi_1 < \theta_2 \).

Since \( \Delta \) is finite we must eventually reach some \( \Psi_n \in \Delta \) with \( \varphi < \theta_1 < \Psi_1 < \theta_2 < \ldots < \theta_n < \Psi_n \). So we have \( \varphi \in |\Delta| \). This completes the proof of Lemma 3.

\[ \varphi \]

**SUBFORMULA THEOREM.** If \( \Pi \) is in normal form then

\[
\begin{align*}
(i) & \quad |\Pi| \subseteq |\Delta, \varphi| \cup |\Delta, \varphi|, \\
(ii) & \quad |\Pi| \cap (|\Delta, \varphi| \setminus |\Delta, \varphi|) \subseteq \Pi.
\end{align*}
\]

**Proof.** By induction on complexity of \( \Pi \).

The basis is obvious. In the inductive step we consider \( \Pi \) by cases, according to the rule of inference last applied. For \( A, R \) and \( I \)-rules, the reasoning is straightforward from IH.

For \( E \)-rules, note that the subproof of the major premiss satisfies the conditions of Lemma 3, so the major premiss is in \( |\Delta| \). Note also that \( \forall E \) and \( \exists E \) discharge only subformulae of their major premisses. With these observations in mind, the detailed verification of (i) and (ii), given IH, is routine.

\[ \varphi \]

**COROLLARY.** In a normal form proof \( \Pi \) the only introduction and elimination rules applied are for logical operators occurring in \( (\Delta, \varphi) \).
THEOREM 1. There is no Proof $\Pi \frac{\{A, \sim A\}}{B}$.

Proof. Such a Proof $\Pi$ could consist of occurrences of only $A, \sim A, \sim \sim A, B, \sim B$ and $\Lambda$, and contain applications of only $\sim I, \sim E$ and $R$. $\sim \sim A$ and $\sim B$ can stand only as assumptions for discharge by $R$. Every occurrence of $A, \sim A$, or $B$ that is not an assumption occurrence must stand immediately below an occurrence of $\Lambda$, as a conclusion of $\sim I$ or $R$. Occurrences of $\Lambda$ must be conclusions of $\sim E$. Consider now the three possibilities of Proof built upwards from $B$:

(i)

\[
\begin{array}{c}
\Pi_1 & \Pi_2 \\
A & \sim A \\
\Lambda & \\
\hline
\end{array}
\]

(ii)

\[
\begin{array}{c}
\Pi_1 & \Pi_2 \\
\sim A & \sim \sim A \\
\Lambda & \\
\hline
\end{array}
\]

(iii)

\[
\begin{array}{c}
\Pi_1 & \Pi_2 \\
B & \sim B \\
\Lambda & \\
\hline
\end{array}
\]

In (i) $\sim A$ must be an assumption occurrence, on pain of abnormality. Thus $\Pi_1$ must be non-trivial, and its conclusion $A$ must be a conclusion of $R$. But then the proof will be abnormal. This rules out (i). In (ii) and (iii), $\Pi_2$ must be trivial. But the assumption $\sim \sim A$ is not discharged by $R$. This rules out (ii). To rule out (iii) observe that in $\Pi_1$, $B$ stands below $\Lambda$ as a conclusion of $R$, contrary to normality.

The following theorem is stated and proved for the classical system $\mathcal{C}$.

THEOREM 2. Given any proof $\Pi$ in normal form, we can effectively find, $\phi$.
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for some \( \Gamma \subseteq \Delta \), either

\[
\begin{align*}
\Gamma \\
(\text{i}) & \quad \text{a Proof } \Sigma, \text{ or} \\
& \quad \Lambda \\
& \quad \Gamma \\
(\text{ii}) & \quad \text{a Proof } \Sigma, \text{ where} \\
& \quad \varphi \\
(a) & \quad \text{if the constellation in } \Pi \text{ whose vertex is the conclusion } \varphi \text{ has} \\
& \quad \text{topmost occurrences of } \varphi \text{ standing as conclusion only of} \\
& \quad \text{eliminations or introductions, then so does the corresponding} \\
& \quad \text{constellation in } \Sigma, \text{ and} \\
(b) & \quad \text{as for (a), deleting "or introductions".}
\end{align*}
\]

Proof. By induction on the complexity of \( \Pi \). The basis is obvious. In the inductive step we consider \( \Pi \) by cases according to the rule of inference last applied. We use the notation

\[
\Pi \rightarrow \Sigma_1 ; \ldots ; \Sigma_n
\]

to mean "Given \( \Pi \), one of the choices \( \Sigma_1, \ldots, \Sigma_n \) can be made to satisfy requirements". Starred proofs mentioned below exist, by inductive hypothesis, as choices of this kind. We suppress mention of premisses wherever possible. We write

\[
\frac{\Pi}{\varphi} (E/I) \quad \frac{\Pi}{\varphi} (E)
\]

to indicate that the terminal \( \varphi \)-constellation in \( \Pi \) satisfies the conditions mentioned in (a) and (b) respectively.

\[
(A) \quad \Pi \rightarrow \Pi^* \\
\Lambda \quad \rightarrow \Lambda \\
\Lambda \quad \rightarrow \Lambda
\]
(R) \[ \frac{\sim A(i)}{\sim A(i)} \]
\[ \Pi \rightarrow \Pi^* ; \Pi^* \]
\[ \frac{\Lambda(i)}{\Lambda} \]
\[ \frac{\Lambda(i)}{\Lambda} \]

(~ I) \[ \frac{\sim A(i)}{\sim A(i)} \]
\[ \Pi \rightarrow \Pi^* ; \Pi^* \]
\[ \frac{\Lambda(i)}{\Lambda} \]
\[ \frac{\Lambda(i)}{\Lambda} \]

(~ E) \[ \frac{\Pi_1(E/I)}{A} \]
\[ \frac{\Pi_2(E)}{\sim A(E)} \]
\[ \Lambda \]
\[ \Lambda \]
\[ \frac{\Lambda}{\Lambda} \]

(& I) \[ \frac{\Pi_1 \Pi_2}{A_1 \& A_2} \]
\[ \frac{\Pi_1^* \Pi_2^*}{A_1 \& A_2} \]
\[ \frac{\Lambda \Lambda}{A_1 \& A_2} \]
\[ \frac{\Lambda \Lambda}{A_1 \& A_2} \]

(& E) \[ \frac{\Pi}{A_1 \& A_2} \]
\[ \frac{\Pi^*}{A_1} \]
\[ \Lambda \]
\[ \frac{\Pi^*}{A_1} \]

(∀ I) \[ \frac{\Pi}{A} \]
\[ \frac{\Pi^*}{\forall x A^a_x} \]
\[ \Lambda \]
\[ \frac{\Pi^*}{\forall x A^a_x} \]

(∀ E) \[ \frac{\Pi}{\forall x A} \]
\[ \frac{\Pi^*}{\forall x A} \]
\[ \frac{\Lambda}{A_i^a} \]
\[ \frac{\Lambda}{A_i^a} \]

(∃) \[ \frac{\Pi}{A_1 \lor A_2} \]
\[ \frac{\Pi^*}{A_i} \]
\[ \Lambda \]
\[ \frac{\Pi^*}{A_i} \]
\[ \Lambda \]
\[ \frac{\Pi^*}{A_i} \]
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\[(\lor E)\]
\[
\begin{array}{c}
\Pi_0^*(E) \\
\Pi_1 \quad \Pi_2 \\
\hline
A_1 \lor A_2
\end{array} \\
\hline
B
\]

\[
\begin{array}{c}
A_1 (i) \\
A_2 (i)
\end{array}
\]

no \( A_i \) no \( A_i \)

\[
B
\]

\[
\begin{array}{c}
\Pi_0^* (E) \\
\Pi_1^* \quad \Pi_2^* \\
\hline
A_1 \lor A_2
\end{array} \\
\hline
B
\]

\[
\begin{array}{c}
A_1 (i) \\
A_2 (i)
\end{array}
\]

\[
\begin{array}{c}
\Pi_1^* \\
\Pi_2^*
\end{array} \\
\hline
\Lambda
\]

\[
\begin{array}{c}
\Pi_0^* (E) \\
\Pi_2^*
\hline
A_1 \lor A_2
\end{array} \\
\hline
B
\]

\[
\begin{array}{c}
A_1 (i) \\
A_2 (i)
\end{array}
\]

\[
\begin{array}{c}
\Pi_1^* \\
\Pi_2^*
\end{array} \\
\hline
\Lambda
\]

where the last two proofs can be normalized without loss of Proofhood by means of the transformation for constellations of type I, if necessary.

\[(\exists I)\]
\[
\begin{array}{c}
\Pi \\
\Pi^* \quad \Pi^*
\end{array} \\
\hline
A_i^* \\
\hline
\exists \alpha A
\]

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(∃E)
\[ \frac{\Pi_1 \rightarrow \Pi_2^* \quad \Pi_2^* \quad \Pi_2^*}{\exists x A(x)B(i)} \]
\[ \frac{\Pi_1^* \quad \Pi_2^* \quad \Pi_2^*}{\exists x A(x)B(i)} \]

COROLLARY 1. If \( \Delta \) is not satisfiable, then there is a Proof \( \Pi' \) for some \( \Delta' \subseteq \Delta \).

\textbf{Proof.} By classical completeness, there is a proof \( \Sigma \) for some \( \Gamma \subseteq \Delta \). By Theorem 2 there is a Proof \( \Pi' \) for some \( \Delta' \subseteq \Gamma \).

COROLLARY 2. If \( \Delta \) is satisfiable, and logically implies \( \varphi \), then there is a \( \Delta' \)

Proof \( \Pi' \) for some \( \Delta' \subseteq \Delta \).

\textbf{Proof.} \( (\Delta, \neg \varphi) \) is not satisfiable. By Corollary 1, and satisfiability of \( \Delta \),

\[ \frac{\Delta', \neg \varphi}{\Pi} \]

there is a Proof \( \Pi' \) for some \( \Delta' \subseteq \Delta \). Apply \( \frac{\Delta', \neg \varphi}{\Pi} \) for the result.'

COROLLARY 3. Suppose we are given Proofs \( \Pi_1, \ldots, \Pi_n \), and

\[ \Delta_1 \quad \Delta_n \quad \varphi_1 \quad \varphi_n \]
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\[ \Gamma, \varphi_1, \ldots, \varphi_n \]
\[ \Sigma \]
\[ \Delta \quad \Delta \]
\[ \Psi \quad \Delta_1 \quad \Delta_n \]
\[ \Pi_1 \ldots \Pi_n \]

Proof. Normalize the proof \( \Gamma, \varphi_1, \varphi_n \) and apply Theorem 2.

6. COMMENTS

Thus for the language \( \sim, \forall, \&, \exists, \forall \) we offer Proof as a satisfactory explication of entailment. All inconsistent sets are Provable so; all consequences of consistent premises are Provable; but the Lewis paradox is not Provable. Transitivity of Proof fails where it least matters: where the new combined premises are inconsistent, by virtue of a Proof effectively determinable from the given Proofs. When transitivity does not fail, the required Proof is likewise effectively determinable from the given Proofs.

By Corollary 2 disjunctive syllogism is Provable. The obvious Proof is

\[
\begin{array}{c}
(1) \quad \sim A \quad \frac{\sim B \quad (1)}{\sim B \quad (2)} \\
A \lor B \quad \frac{\Lambda \quad (1)}{\Lambda} \\
\frac{\Lambda \quad (2)}{B}
\end{array}
\]

This is welcome to anyone concerned with the formalization of mathematical reasoning involving axioms of dichotomy, trichotomy etc. Although by Corollary 2 we would also expect \( B \lor \sim B \) to be Deducible from \( \{A\} \), reasoning similar to that for Theorem 2 shows that \( B \lor \sim B \) is Deducible only from the empty subset of \( \{A\} \); no Proof with conclusion \( B \lor \sim B \) has any undischarged assumptions of the form \( A \).

We have regarded entailment as a subrelation of deducibility – essentially as a meta-relation between sentences of the object language. We have no (iterable) connective in the object language to be read as “entails”. The
reader will easily discover for himself how the presence of \( \supset \) with its usual introduction and elimination rules blocks the proof of Theorem 2 above. But perhaps a connective arrow could be introduced into the object language, with suitable constraints on its introduction rule, so that Theorem 2 remains intact. It remains to investigate this possibility. Yet we must emphasize that in the absence of such a connective we suffer no loss in the expressive and deductive capacity required for first order classical mathematics. Everything we want proofs to do can be done by Proofs. The Lewis paradoxes are excised with minimum mutilation to the deductive fabric of mathematics. Moreover the effective methods of normalization and extraction implicit in Corollary 3 show that Proofs of theorems from axioms are related to Proofs of lemmata from axioms and theorems from lemmata in what Anderson and Belnap would call a "heartwarming" way (p. 299).

In closing, a few remarks would perhaps be in order concerning the closest precursor of my system. Smiley's investigation of entailment as a relation between premises and conclusion is perhaps the closest in spirit to mine. Direct comparison, however, is difficult for two reasons. First, he treats sequences and not sets of premises (and so speaks of permutations of premises, and of deletions of repetitions). Secondly, he takes \( \supset \) as primitive (while not reading it as "entails"!), precisely the connective which must be absent for my "Extraction" Theorem 2 to hold. Insofar as Smiley approaches entailment via deducibility (in his third and fourth proposed systems) he concerns himself with global properties of the deducibility relation (many dependent on sequencing of premises) and with the rather crude conventional notion of linear proof. Moreover he preserves unrestricted transitivity for his entailment relations. By contrast, the emphasis in the present paper has been on the internal structure of proofs designed to capture entailments. The result is an entailment relation which fails to be transitive, as already remarked, where it least matters. In this connection it is interesting to consider some of Smiley's general remarks about transitivity (p. 242):

The need for an unrestrictedly transitive entailment-relation for serious logical work is no reason at all against accepting a relation which is not unrestrictedly transitive as being a satisfactory reconstruction of an intuitive idea of entailment. But the need itself is undeniable: the whole point of logic as an instrument, and the way in which it brings us new knowledge, lies in the contrast between the transitivity of "entails" and the non-transitivity of "obviously entails", and all this is lost if transitivity cannot
be relied on. Of course, if there is an effective way of predicting when transitivity will hold then most of the objection vanishes; ... but I do not even begin to see how the thing might be done in, say, predicate logic ...

What our Corollaries above show, however, is that not all is lost if transitivity cannot be relied upon. When we “put together” Proofs, normalize and extract, what we get is either a Proof of the final conclusion, or a Proof of \( \Lambda \) from the combined premises. Nothing more could be desired.

Smiley also attempted a semantical definition of an entailment relation (p. 240):

\[ A_1, \ldots, A_n \vdash B \text{ if and only if the implication } (A_1 \land \ldots \land A_n) \supset B \text{ is a substitution instance of a tautology } (A_1' \land \ldots \land A_n') \supset B', \text{ such that neither } \vdash B' \text{ nor } \vdash \sim (A_1' \land \ldots \land A_n'). \]

Now we know that there are Proofs of conclusions other than \( \Lambda \) from inconsistent sets of premises: for example, \[ \frac{A \& \sim A}{A} \]. Note, however, that the latter is a substitution instance of the Proof \[ \frac{A \& B}{A} \] whose premiss (set) is consistent. I conjecture that this holds in general:

\[ \Delta \]

Every Proof \( \Pi \) of \( \varphi (\neq \Lambda) \) from inconsistent \( \Delta \) is a \( \varphi \)

\[ \Delta' \]

substitution instance of some Proof \( \Pi' \) with \( \Delta' \) consistent. \( \varphi' \)

In other words, in Proving \( \varphi \) from \( \Delta \) we do not “trade illicitly” on the inconsistency of \( \Delta \), but can make do with as little logical form in the members of \( \Delta \) as fail to expose the inconsistency.

This corresponds to the non-validity of \( \sim (A_1' \land \ldots \land A_n') \) in Smiley’s semantical definition. What about the non-validity of \( B' \)? That is, what if the following conjecture held:

\[ \Delta \]

Every Proof \( \Pi \) of valid \( \varphi \) from non-empty \( \Delta \) is a \( \varphi \)

\[ \Delta' \]

substitution instance of some Proof \( \Pi' \) with \( \varphi' \) non-valid ...?
Substitution preserves validity and inconsistency. Thus our notion of Proof would exactly characterize that extension of Smiley's semantical relation in which also every inconsistent set entails $\Lambda$, and the empty set entails each logical truth.

But the second conjecture is false. A counterexample is the following Proof of the valid formula $\sim (\sim (A \& \sim B) \& (A \& \sim B))$ from the premiss $\sim (B \& \sim A)$:

$$
\begin{align*}
\sim (A \& \sim B) \& (A \& \sim B) & \quad (4) \\
A \& \sim B \quad & \quad (1) \\
A \quad & \quad (1) \\
A \& \sim B \\
A \quad & \quad (2) \\
\sim A \\
B \& \sim A \\
(B \& \sim A) & \quad (B \& \sim A)
\end{align*}
$$

In this proof we note that the step marked (1) is an application of classical reductio rather than a quodlibet by virtue of formally discharging the assumption $\sim B$. But this assumption seems to have been "smuggled" spuriously into the proof by virtue of what one might call a "minimal" occurrence of $A$ in the third line. It stands, namely, as the conclusion of an elimination and as a premiss of the corresponding introduction, where the major premiss of that elimination and the conclusion of that introduction are occurrences of the same formula.

Perhaps one could define an even stronger notion of normal form designed to exclude minimal occurrences of formulae as well as maximal ones. If our normalization and extraction theorems then remain untouched, it might turn out that the second conjecture above held, thereby clinching
adequacy of Proof for Smiley's semantical relation. That, however, is a topic for future research.

University of Edinburgh

NOTE

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