I propose an anti-realist account of truth and paradox according to which the logico-semantic paradoxes are not genuine inconsistencies. The ‘global’ proofs of absurdity associated with these paradoxes cannot be brought into normal form. The account combines epistemicism about truth with a proof-theoretic diagnosis of paradoxicality. The aim is to combine a substantive philosophical account of truth with a more rigorous and technical diagnosis of the source of paradox for further consideration by logicians. Core Logic plays a central role in the account on offer. It is shown that the account is not prey to the problem of revenge paradox.
Don’t say: “There must be something common, or they would not be called ‘paradoxes’” — but look and see whether there is anything common to all. — For if you look at them you will not see something that is common to all, but similarities, relationships, and a whole series of them at that. To repeat: don’t think, but look!
— with apologies to Wittgenstein, *Philosophical Investigations*, section 66

1 On the aims of this study

As pointed out by Horsten 1995, philosophers have tended to debate the nature of truth without much regard for the problems posed by the logical and semantic paradoxes; and logicians have tended to seek novel solutions to these problems without much regard for those questions about the nature of truth that interest philosophers. The former, philosophical, kind of work has been more ruminative, arguing about the relative merits of correspondence, coherence, pragmatic, epistemic and deflationist theories of truth (to mention just a few of the main theories). The latter, logical, kind of work is perforce more technical. It investigates a variety of logical approaches (to semantically closed languages) that have earned the monikers ‘gap’, ‘glut’, ‘paracomplete’ or ‘paraconsistent’ (again, to mention just a few of the main ones).

With a few notable exceptions, workers have tended to stay within one or other of these camps. As Scharp (2013, chapter one) makes clear in his admirably thorough survey of the terrain, what is needed is an account of truth that both explains its nature and resolves the paradoxes. The account needs to be both philosophically substantive and logically sophisticated.

Scharp provides a very useful classification of the few extant accounts that can make some claim to be pursuing both these goals, and hence deserve to be called ‘unified accounts’ of truth and paradox. These are the accounts to be found in Barwise and Etchemendy 1987, McGee 1991, Gupta and Belnap 1993, Soames 1999, Maudlin 2004, Field 2008 and Beall 2009. Scharp also has his own diagnosis to offer of (what he takes to be) the fundamental problem with truth that manifests itself in the paradoxes — to wit, that truth is what he calls an *inconsistent concept*. And he offers a pair of replacement concepts, which he calls *ascending truth* and *descending truth*, in an attempt to avoid the paradoxes, developing his own variant of a ‘re-

\[1\] Scharp also lists Horwich in this connection, but relies on exegesis by others to attribute epistemicism to him, and notes anyway that it is not at all clear just what logical account Horwich might offer for the paradoxes. Hence I am not listing Horwich among these other figures here.
vision theory’ for descending truth. I do not aim here to discuss the merits of Scharp’s positive proposal. I should, at this early stage, however, let the reader know that my own approach to the paradoxes does not allow that any inconsistency has been demonstrated, in the case of any logico-semantic paradox. The reasons available to back such a view are quite subtle, and will be sketched below.

The aim of this study is to offer a brief sketch of a ‘unified’ approach that derives from my earlier work on the (then separately treated) topics of (i) semantic anti-realism, and (ii) a proof-theoretic approach to paradox. To the best of my knowledge, there is no fully developed ‘anti-realist line’ concerning the paradoxes. Certainly Dummett, who initiated semantic anti-realism, appears to have confined his remarks about paradoxes to Wang’s Paradox (which has to do with the problem of vagueness, not the problems of semantic closure). So this study might be interpreted as an attempt to work out such an anti-realist line, as I am in broad sympathy with Dummett’s views on epistemically constrained truth, the invalidity of the Principle of Bivalence, and the need for logical reform as a consequence.

I hasten to emphasize the tentative and conjectural character of much of what is to follow. It is a first attempt to sketch and develop some of the main features of a reflective equilibrium that I believe is available to the anti-realist who is minded to engage with the problems posed by the well-known semantic paradoxes. There may well be other thinkers of a broadly anti-realist persuasion who will demur from some of the important details of the position about to be outlined. If they are hereby provoked into criticizing this position and offering an alternative, then this study will have achieved another good part of its purpose. Its residual purpose, for those who by and large concur with the position broadly outlined here, is to provoke interest in the research project, of a more formal nature, that needs to be carried out in order to establish with full logical rigour such relevant technical results as would shore up the philosophical position on offer. But this study will also have achieved a measure of success if it provokes any readers to undertake the necessary research on formal matters that could undermine some of the philosophical and methodological conjectures put forward below.

We are exploring a territory in which no one has so far established an impregnable position, in which their philosophical ‘take’ is fleshed out with faithfully formalized theses furnished with proofs so as to become dispositive metatheorems. Rather, the range of positions is so diverse (as witnessed by Scharp’s own classificatory study) that expertise in this area can fairly be said to consist more in knowing about all the ‘ins and outs’ of competing positions, and their mutual criticisms, than it does in working out a single,
sustained account addressing all the philosophical issues, and buttressing it by appropriate metatheorems. Such a finished reflective equilibrium on the problem(s) of truth and paradox in semantically closed languages appears, in light of continuing disputes among the experts, to have eluded even the best minds in the history of the subject. Paradox remains—along with time-honoured topics such as the existence of God, free will, consciousness, and the objectivity of value—one of the most devilishly elusive, ideologically divisive, and hotly contested areas of philosophical inquiry. I do not pretend to be offering a ‘last word’ on the topic. Rather, I am here just picking up the gauntlet of an imaginary critic of anti-realism, who is seeking to find weaknesses in the anti-realist’s global position, and who happens to inquire, in a sceptical tone, ‘What’s the anti-realists’ line on the semantic paradoxes? Dummett said very little about the topic. Might that have been because anti-realists cannot handle it?’

Well, we shall have to look and see.

2 A point about terminology

We are accustomed to speaking loosely about the reasoning involved in paradoxes. We say that paradoxes are ‘established’, or ‘proved’, or ‘demonstrated’, or ‘generated’, or ‘shown to arise’ . . . . The variety of terms can be misleading for the philosophical logician whose job it is to introduce a certain degree of rigour in discussions of these complex issues. For the purposes of this discussion, I shall reserve the verb ‘generate’ for the informal reasoning that makes one aware that one is dealing with a paradox. Typically, it involves two pieces of informal argumentation: one for the conclusion that the paradoxical sentence in question is true; and the other for the conclusion that it is false. Alternatively, the former piece of reasoning can be for the sentence itself as its conclusion; and the latter can be a reductio ad absurdum of the sentence itself as an assumption.

Typically, as soon as one is in possession of two such pieces of reasoning, one realizes that something has gone terribly wrong. A paradox has been generated. For it looks as though one can now construct a ‘global disproof’ or ‘global reductio’ by suitably assembling the two pieces of reasoning already in hand, and thereby deriving absurdity—not from the sentence itself, but from the ‘framework of principles’, or ‘set of background assumptions’, or whatever it is that one has somehow been taking for granted while the paradox was being generated. In some striking cases, the absurdity seems to arise ‘out of thin air’, there being no explicit culprit assumptions ‘in the
background’, that one can assemble for collective reproach.

That, however, is different from saying that an inconsistency has been established or proved. I reserve these more exigent terms for situations where the result of putting the two pieces of reasoning together is itself in good order, from a proof-theoretically informed point of view to be explained here. I must caution the reader in advance that, for example, the Liar paradox does not (on the account to be ventured here) establish or prove any inconsistency (either in our language, or in our system of concepts, or among our theoretical beliefs). But of course I readily own that the Liar succeeds in generating a sense of paradox. It has to; for otherwise we would not pay it the attention it so richly deserves.

My main contention is that every semantic paradox is like the Liar in this main regard: while it may generate a sense of paradox, via informal arguments purporting to establish both the truth and the falsity of certain sentence(s), these arguments nevertheless do not succeed in establishing any genuine global inconsistency.

For this very reason, Russell’s so-called ‘paradox’ in set theory is no paradox at all. Upon proper analysis of the language and system of concepts in which it arises, it can, qua purported paradox, be laid entirely to rest. The informal reasoning associated with it can be regimented, as a perfectly good derivation of \( \bot \) (absurdity) from the assumption that there is such a set as the set of all and only those sets that are not members of themselves. This derivation is a normal disproof, that is, a proof of \( \bot \) (from a set of premisses) that is in what proof theorists call normal form. It is, moreover, a constructive and relevant proof. The reasoning involved in Russell’s Paradox, upon closer proof-theoretic analysis, establishes a negative existential theorem of set theory: \( \neg \exists y \ y = \{ x | x \notin x \} \).

3 Classification of unified approaches

Scharp provides an interesting ‘three-dimensional’ classification of unified approaches to truth and paradox. The dimensions are:

(i) a philosophical account of the nature of truth;

(ii) a diagnosis of the source of paradox, according to one’s preferred semantics for natural language; and

(iii) an account of one’s preferred logic that combines various logical principles with well-chosen aletheic principles.
Our approach combines the anti-realist account of the nature of truth from Tennant 1987a and 1997 with the proof-theoretic account of the paradoxes first given in Tennant 1982 and re-visited in Tennant 1995. As far as (iii) is concerned, I hasten to assure the reader that the alethic principles employed in my account are the introduction and elimination rules for the (monadic) truth predicate:

\[
\frac{\varphi}{T^\varphi} \quad \frac{T^\varphi}{\varphi}
\]

Note the use of the logician’s corner-quotes. The anti-realist strand of my overall position contributes two components that are worth stating separately (as (1) and (2) below), and which combine with the proof-theoretic account of paradox ((3) below), to fix positions on each of Scharp’s three classificatory dimensions.

1. We have epistemicism about truth (Scharp’s terminology) as the philosophical account of its nature;

2. We have indeterminacy of truth-value (Scharp’s terminology again) as the diagnosis of the source of paradox according to one’s preferred semantics for natural language;

3. We have a proof-theoretic account of the peculiar logic of paradox (my own, perforce ungainly, terminology).\(^2\) This account stresses the non-normalizability of the purported constructive global disproofs associated with the paradoxes. It also allows one to treat the truth-predicate as governed by the obvious introduction and elimination rules stated above.

The combination of these three components fills an inviting niche in the spectrum of possible unified accounts, on Scharp’s aforementioned classification of them. My proof-theoretic diagnosis of paradoxicality provides a way to demur from Scharp’s conclusion that truth is an inconsistent concept. It also allows one to refrain from making any bizarre or recondite logical reforms in order to be able to handle the paradoxes. Perhaps most agreeably, the new account presents its logic as a transparent system of rules of inference, and does not resort to infinitistic semantic constructions. Most certainly — and attractively — it does not present the logic in semantical terms only, leaving

\(^2\)Here I intend 'peculiar' in both senses — there is something peculiar to the (constructive) proofs associated with paradoxes, and something peculiar about them. This was discovered not by thinking, but by looking.
one in the dark as to how the logic would actually work at the inferential level.

With regard to dimension (iii), and my choice (3) above, Scharp himself has suggested (private communication) that the logical account should go under his ‘paraconsistent’ heading; but on this I demur. Paraconsistentists are concerned, once they have legitimately derived absurdity, to prevent that from ‘exploding’ into a logical commitment to any consequence whatsoever. Now, while the logic I recommend is indeed a paraconsistent one—in that it does not contain the first Lewis Paradox $A, \neg A : \neg B^3$—it turns out that I am not proposing to exploit this feature of the logic in order to handle (or avoid) the problem of paradox. Rather (as the reader will become aware) I am arguing that we are not dealing with ‘legitimately derived’ absurdities at all. Details will emerge below.

My diagnosis of paradox is a specifically anti-realist one. I propose to characterize paradoxes by adverting to the non-normalizability of the purported constructive global disproofs associated with them. (I shall in due course be illustrating this phenomenon in some detail, by reference to the paradox of The Liar.) The anti-realist’s ability to identify this criterial feature of paradoxicality is thanks to her constructive logic, with its rejection of strictly classical rules of negation, and its concomitant stress on how warrants for assertion must in principle be able to be cast in normal form. As we shall see below, if one has recourse to the classical rule (CR) of reductio ad absurdum, one can mask the true nature of paradox that the anti-realist is at pains to make clear. This provides yet another arrow for the anti-realist’s quiver in the ongoing debate with the realist over which logic is the right logic. If the conclusions and conjectures of this study are correct, then the anti-realist is able to get to the bottom of what is going on with paradoxes, in a way to which the realist (who uses classical logic) can be blind. Of course, it is always open to the realist (for the purposes of concurring with the analysis of paradoxicality here proposed) to temporarily ‘don the anti-realist’s hat’, by restricting himself to the constructive logic of the anti-realist, with the limited purpose of being able to display the true nature of the paradoxes. But that is a potentially self-undermining indulgence on the part of the realist. After appropriating one’s opponent’s logic for certain restricted purposes, one is bound to be faced with the challenge of explaining why the same tidy analysis is not afforded within the framework of all the primitive rules that one regards, oneself, as valid.

---

3Nor, indeed, does it contain the ‘negative’ version of the first Lewis Paradox, namely $A, \neg A : \neg B$. In this respect it differs from the system of Minimal Logic in Johansson 1936.
Perhaps one of the most striking features of the unified account on offer here is that the motivation for its various components has an underlying unity. The Dummettian anti-realist insists on the manifestability of grasp of meaning. There is an ensuing stress on the importance both of the harmony of the inference rules governing logical operators, and of canonical warrants as justifications of assertions. This has led to a full-fledged inferentialism about logic and mathematics, along with recommendations for logical reform. This same inferentialism, with its rule- and proof-theoretic methodology, upon addressing the issue of paradox in a semantically closed language, offers a crucial insight into the nature of the ‘global disproofs’ associated with the semantic paradoxes. By ‘following the meaning-theoretic money’ we are able to reach at least the beginnings of a potentially fruitful, unified account of truth and paradox on the basis of that underlying methodology.

4 A new unified account of truth and paradox

4.1 The epistemicist component

For the anti-realist, every truth is knowable, and its truth consists in the existence of a(n in principle) surveyable truthmaker, also called a (canonical) proof, or warrant. When a speaker sincerely asserts a declarative sentence, the listener is given to understand that the speaker has a warrant to back the assertion, or at least possesses an effective method for finding such a warrant, if one is called for. Also — as needs to be emphasized — the listener is given to understand that the speaker knows of no countervailing or undercutting or defeating construction or considerations that would diminish or cancel the justificatory force of the proof on offer. In mathematics, for example, the default consensus is that proof is dispositive. For any mathematical sentence, if one has a proof of it, then one cannot also have a disproof of it.

Although the symmetry is seldom remarked on, one can of course say correlative things about falsity, from the anti-realist’s point of view. Every falsehood is knowable as such, and its falsity consists in the existence of a(n in principle) surveyable falsity-maker. (See Tennant 2010.) When a falsity-

4Truthmakers are not in general surveyable. The formal ones characterized in Tennant 2010 and 20XX, for example, take the mathematical form of trees whose branches are finitely long, but in which some of the nodes (corresponding to steps that falsify existentials, or that verify universals) sprout domain-many branches, one for each individual in the domain. So with an infinite domain, these formal truthmakers can be ‘infinitely wide’, hence not surveyable.
maker is surveyable, it is called a (canonical) disproof (that is, a proof of absurdity as its conclusion, and having the sentence in question—whose falsehood is thereby established—as an undischarged assumption). When a speaker sincerely rejects a declarative sentence, the listener is given to understand that the speaker has a disproof to back the rejection, or at least possesses an effective method for finding such a disproof, if one is called for. Also—as needs similarly to be emphasized—the listener is given to understand that the speaker knows of no countervailing or undercutting or defeating construction or considerations that would diminish or cancel the justificatory force of the disproof on offer. In mathematics, for example, the default consensus is that disproof is dispositive. For any mathematical sentence, if one has a disproof of it, then one cannot also have a proof of it.

The association of a truth-value with a sentence—True or False—cannot, therefore, transcend our ability, in principle, to come to know it, when it obtains. The anti-realist will not be drawn into asserting, of any given sentence \( \varphi \), that either \( \varphi \) is true or \( \varphi \) is false (i.e., not true). The anti-realist is prepared to assert that disjunction—that instance of the Principle of Bivalence—only when apprised of which disjunct to assert.

Moreover, when one of the two cases is realized, the (canonical) proof or disproof in question must, according to the anti-realist (who espouses a constructive logic), be in normal form. Or, if it is not (yet) in normal form, then it must be the case that it could at least be normalized, that is, could in principle be brought into normal form by applying suitable reduction procedures only finitely many times. This is of overriding importance, from the anti-realist’s point of view. The construction backing the speech act—the proof backing the assertion or the disproof backing the rejection—must trace the grounded reasons why it reaches the conclusion that it does. This is possible only when the constructions in question are able to be brought into normal form. The following principle is a cornerstone of proof-theoretic foundations for constructive mathematics:

For every proof \( \Pi \) that we may provide for a mathematical theorem \( \varphi \), it must be possible, in principle, to transform \( \Pi \), via a finite sequence of applicable reduction procedures, into a canonical proof of \( \varphi \), that is, a proof of \( \varphi \) that is in normal form, so that none of the reduction procedures is applicable to it.

The qualifier ‘in principle’ here is doing a lot of work. This principle of ‘canonicalizability’ is actually honoured in the breach, in so far as the vast majority of proofs that constructive mathematicians accept, and take as justification for the assertion of their mathematical theorems, are indirect
proofs that have not been fully normalized. (Nor, indeed, are they at all likely to have been fully formalized!) This is because they are typically accumulations of proofs — proof of lemmas from axioms, followed by further proof of the sought theorem from the lemmas, plus possibly further axioms. The lemmas are, in effect, being elided by cuts. It is the exception rather than the rule that the constructive mathematician would actually bother to transform the ‘final’ proof that results from such accumulations, or cuts, into a single streamlined, ‘cut-free’ proof of the theorem from the axioms. For we know that the process of normalizing a proof (equivalently: eliminating the cuts) leads, in general, to hyper-exponential blow-up in the length of proof. In fact, what makes deductive progress in mathematics feasible, quite apart from the brevities afforded by highly compiled definitions of mathematical concepts in terms of primitives, is the hyper-logarithmic reduction in length of proof afforded by the practice of liberally interpolating lemmas, and accordingly performing cuts as necessary.

The verb ‘perform’ here deserves further comment. In Gentzen’s original systems, one can literally perform a cut, by applying the Cut Rule of the sequent calculus:

\[
\Delta : \lambda, \lambda, \Gamma : \varphi \\
\Delta, \Gamma : \varphi
\]

— or, in the case of natural deductions, by grafting a copy of one’s proof \( \Pi \) of the cut sentence \( \lambda \) on top of each of \( \lambda \)'s premiss-occurrences in the proof \( \Sigma \) of one’s sought conclusion \( \varphi \):

\[
\Delta \\
\Pi \\
(\lambda), \Gamma \\
\Sigma \\
\varphi
\]

But Gentzen’s calculi thereby allow for irrelevance. In alternative relevantized calculi, such as the system of Core Logic (or its classicized extension in Tennant 2012b and Tennant 2013a) one does not ‘perform’ cuts in this way, because the cut rule is not a primitive (or derivable) rule of the system. All core proofs are in normal form. But cut is an admissible rule, and is so in a way that can lead to epistemic gain. The following metatheorem holds for classicized Core Logic, hence for Core Logic itself. The result is proved in detail in Tennant 2013a.

There is an effective method \([ , ]\) that transforms any two core
proofs

\[ \Delta \vdash \lambda \Gamma \]
\[ \Pi \vdash \Sigma \]
(\text{where} \( \lambda \notin \Gamma \) \text{and} \( \Gamma \) may be empty)
\[ \lambda \vdash \varphi \]

into a core proof \([\Pi \Sigma]\) \text{of} \( \varphi \) \text{or of} \( \bot \) \text{from (some subset of)} \( \Delta \cup \Gamma \).

It is the judicious interpolation of lemmas to be cut that breaks an otherwise impossibly long deductive odyssey into manageable segments that can be strung together into a negotiable overall route.

Most of this is subconscious wisdom among mathematicians who are innocent of modern proof-theoretic analysis of the foundations of their discipline. But it is crucial to bear this much in mind: we are prone to expect, and accept, passages of reasoning that take the form (in, say, the simplest of cases, which is almost a chain, but which is complex enough to illustrate the point at issue):

\[ \alpha_1, \text{therefore} \ldots \lambda_1; \text{but} \alpha_2, \text{therefore} \ldots \lambda_2; \text{but} \ldots \lambda_{n-1}; \text{but} \alpha_n, \text{therefore} \ldots \varphi. \]

Graphically:

\[
\begin{array}{c}
\alpha_1 \\
\Pi_1 \\
\lambda_1, \alpha_2 \\
\Pi_2 \\
\lambda_2 \\
\vdots \\
\Pi_{n-1} \\
\lambda_{n-1}, \alpha_n \\
\Pi_n \\
\varphi
\end{array}
\]

Here, the lemmas \( \lambda_k \) \((1 \leq k < n)\) are seen to depend, ultimately, on the axioms \( \alpha_1, \ldots, \alpha_k \); and the theorem \( \varphi \) is seen to depend, ultimately, on the axioms \( \alpha_1, \ldots, \alpha_n \). Each lemma \( \lambda_i \) is a cut sentence: it is the conclusion of the proof \( \Pi_i \) above it, and a premiss of the proof \( \Pi_{i+1} \) below it. With well-chosen lemmas \( \lambda_1, \ldots, \lambda_{n-1} \), the sum of the lengths of the proofs \( \Pi_1, \ldots, \Pi_n \) is manageable. But, if one eliminates cuts so as to produce a \textit{direct} (because
normal, or cut-free) proof $\Pi$ of the theorem $\varphi$ from the axioms $\alpha_1, \ldots, \alpha_n$:

$$\begin{array}{c}
\alpha_1, \ldots, \alpha_n \\
\Pi \\
\varphi
\end{array}$$

then the length of $\Pi$ might well be unmanageable.

The lesson to keep in mind from this pragmatically enforced resort to frequent interpolation of lemmas is this: we are accustomed to attaining deductive results that themselves rest upon intermediate deductive results. We never seriously entertain the possible truth of a mathematical (or scientific) claim that is contrary to, or the contradictory of, any result we have thus proved. The deep reason for this is the \textit{in-principle-normalizability} of whatever (indirect)$^5$ proof we might have discovered for a claim we are accordingly prepared to assert.

This habit of mind — this clutch of expectations — does not readily lose its grip when we venture into the more dangerous territory of the logical and semantic paradoxes. And therein, says the anti-realist, lies the problem. We must cease to \textit{think} in our customary (and however rigorous) ways; instead, we must \textit{look}.

4.2 \textit{The indeterminacy component}

Paradoxes like the Liar are prime candidates for the status of being ‘exceptions’ to the Principle of Bivalence. But, notoriously, the anti-realist has to be very careful indeed not to express her refusal to accept Bivalence in the form of a claim to the effect that there are counterexamples to it. Rather, she should go no further than to assert

$$\neg \forall \varphi (T \langle \varphi \rangle \lor \neg T \langle \varphi \rangle).$$

The anti-realist, who does not accept full classical logic, does not allow that this entails

$$\exists \varphi \neg (T \langle \varphi \rangle \lor \neg T \langle \varphi \rangle).$$

And it is important to avoid commitment to the latter, since it is \textit{intuitionistically} inconsistent, as the following lemma makes clear.

---

$^5$The reader should bear in mind that by ‘indirect proof’ here I mean only a (constructive) proof that involves a subproof of a lemma as its conclusion, and a subproof that uses that lemma as a premiss. I do not mean a proof that appeals to classical \textit{reductio ad absurdum}. Proofs of the latter kind are often also called indirect, but for different reasons. The indirect proofs to which I am referring here are all constructive. They contrast with direct or \textit{canonical} proofs in the sense of Prawitz 1977 (at p. 22).
Lemma 1. \( \neg(\theta \lor \neg\theta) \vdash I \bot. \)

Proof (in normal form):

\[
\begin{align*}
\neg(\theta \lor \neg\theta) & \quad \vdash \neg\theta \\
\theta \lor \neg\theta & \quad \vdash (1) \\
\neg(\theta \lor \neg\theta) & \quad \vdash \bot \\
\end{align*}
\]

The safest way for the anti-realist to express her refusal to accept Bivalence, in the eyes of a critic who thinks that to deny the principle is to sail a little too close to the wind, is simply to refrain from asserting it.

In the context of semantically closed languages and the problem of paradox, however, it might appear that the anti-realist can be a little more committal about possible exceptions to Bivalence. Let us define the determinacy predicate \( D \) as follows:

\[
D\sigma \equiv T\sigma \lor \neg T\sigma,
\]

where \( \sigma \) is any sentence-denoting term. Now consider the Liar sentence \( \ell \), namely \( \neg T\ell \). Usually the reader is invited to consider the following display:

\[
(\ell) \quad \neg T\ell.
\]

Because of the logico-grammatical form of the Liar sentence, we have the following axiom in our semantically closed language. All parties to the discussion of the Liar Paradox have to agree that this axiom is unimpeachable; it is a precondition for considering \( \ell \) to be the Liar Paradox:

\[
\ell = '\neg T\ell'.
\]

Note that ‘\( \ell \)’ is a (syntactically simple) name of the Liar Sentence, which sentence involves that very name itself. The Liar sentence of course has another, more complex, name, namely ‘‘\( \neg T\ell \)’’. More on that anon. By writing down ‘\( \ell \)’, one is not thereby supposing or asserting; rather, one is referring (to the Liar sentence). And the Liar sentence, by containing the name ‘\( \ell \)’, refers to itself. It is the sentence (of our semantically closed formal language) that says that \( \ell \) is not true. The axiom above states what \( \ell \) is. Hence the need for logical quotation marks on the right-hand side of
that identity statement, forming the complex name mentioned above. \( \ell \) is (identical to) the sentence consisting of the negation sign ‘\( \neg \)’, followed by the truth predicate ‘\( T \)’, followed by the name ‘\( \ell \)’.

**Lemma 2** \( \neg D\ell \), i.e. \( \neg (T\ell \lor \neg T\ell) \).

*Proof (in normal form):*

\[
\begin{array}{c}
\ell = \neg T\ell \\
\neg T\ell
\end{array}
\]

\[
\begin{array}{c}
T\ell \\
\neg T\ell
\end{array}
\]

\[
\begin{array}{c}
\ell = \neg T\ell \\
\neg T\ell
\end{array}
\]

The reader will now raise an objection: Lemma 2 purports to establish a result that is of a logical form that Lemma 1 shows to be intuitionistically inconsistent. Has not the anti-realist thereby contradicted herself? The surprisingly negative answer (see Sect. 4.3.5) will have to await further explanation of some important proof-theoretical ideas.

4.3 The proof-theoretic component

4.3.1 Standard ideas from proof theory

I have just adverted to the proof-theorist’s notions of reduction procedure, normal form, and normalization. For the purposes of this study, I do not intend to explain all of these in the limited space available. The reader will find a wealth of explanation in the *loci classici* Gentzen 1934, 1935 and Prawitz 1965, and discussion of the philosophical applications of these proof-theoretic ideas in Prawitz 1974, Dummett 1991 and Tennant 1997. Here, I shall be concerned only with the reduction procedures that will be needed for a discussion of certain standard paradoxes. There are really only
two such procedures: Negation-Reduction, and Avoiding Prolixity.

\[\neg \text{-Reduction:} \quad \Gamma, \Sigma \vdash \varphi \quad \vdash (i) \quad \Pi, \Gamma \vdash \neg \varphi \quad \varphi \quad \Pi, \Gamma, \Sigma \vdash \varphi \quad \vdash (i) \quad \Gamma, \Sigma \vdash \varphi \]

\[\text{Avoiding Prolixity:} \quad \Delta, \Theta \vdash \varphi \quad \Pi, \Gamma, \Sigma \vdash \varphi \]

These ideas were first applied to the problems of paradox in Tennant 1982. It was shown there that the global disproofs (i.e. proofs of absurdity) associated with all the standard semantic paradoxes (the Liar Paradox; Grelling’s Paradox; Curry’s Paradox; and Tarski’s Quotational Paradox) are not normalizable. The same is true of the Postcard Paradox, which was not treated in that paper. Later, in Tennant 1995, it was shown that the same holds true for Yablo’s paradox (for which, see Yablo 1993). The interesting feature was that, whereas the non-normalizability of the global proofs of absurdity associated with any of the standard paradoxes arises from the ‘looping’ of the reduction sequence, the non-normalizability of the global proof of absurdity associated with Yablo’s paradox is owed to a kind of ‘spiralling to infinity’ that is easy to characterize. It is the same symptom of paradox in each case — the non-termination of any reduction sequence that one might embark upon in a (vain) attempt to turn the global disproof in question into normal form.

4.3.2 Our proof-theoretic diagnosis of the Liar

It is instructive to see how these ‘pattern-detecting’ ideas apply in the simplest case of paradox, namely the Liar. Remember we have the axiom

\[\ell \equiv \neg T\ell\]
and the rules of $T$-Intro and $T$-Elim:

$$
\frac{\varphi}{T \varphi} \quad \frac{T \varphi}{\varphi}.
$$

**Lemma 3 (The Liar is Provable) $\vdash \neg T \varphi$.**

*Proof (in normal form):*

$$
\Pi : \begin{array}{c}
(1) \\
T \varphi \\
\ell = \neg T \varphi
\end{array}
\frac{T \neg T \varphi}{\neg T \varphi} \\
(1)
\frac{T \neg T \varphi}{\neg T \varphi} \\
\frac{\bot}{\bot}
$$

**Lemma 4 (The Liar is Refutable) $\neg T \varphi \vdash \bot$.**

*Proof (in normal form):*

$$
\Sigma : \begin{array}{c}
T \neg T \varphi \\
\ell = \neg T \varphi
\end{array}
\frac{T \neg T \varphi}{\neg T \varphi} \\
\frac{T \neg T \varphi}{\neg T \varphi} \\
\frac{T \varphi}{\bot}
$$

So: we have a normal-form *proof*, $\Pi$, of The Liar ($\neg T \varphi$); and we have a normal-form *disproof*, $\Sigma$, of The Liar. Will this not precipitate a crisis of global inconsistency, upon accumulating these two proofs?:

$$
\emptyset \\
\Pi \\
\Lambda : \begin{array}{c}
(\neg T \varphi)
\end{array}
\Sigma \\
\frac{\bot}{\bot}
$$

*The answer is negative. We need to conduct a normalizability check.* Normally we would not bother to do so. But we are not doing mathematics here, nor are we testing empirical scientific theories formulated in a semantically
open language. We are instead reasoning about a semantic paradox, in a semantically closed language; so we need to double-check everything as we go along.

As can be expected with the results of accumulations (or cuts) in general, this ‘global disproof’ (Λ) is not in normal form. This is because in Π the concluding sentence $\neg T\ell$ stands as the conclusion of an application of $\neg$-I, while in Σ it stands as the major premiss of an application of $\neg$-E. So $\neg$-Reduction is applicable. Note that $\neg T\ell$ enjoys two premiss-occurrences in Σ. The would-be global disproof (Λ) is therefore

$$
\begin{array}{c}
\Pi \\
\Pi
\hline
\neg T\ell
\end{array}
\quad
\begin{array}{c}
\frac{T \neg T\ell \quad \ell = \neg T\ell}{\bot}
\end{array}
$$

which I shall abbreviate as

$$
\begin{array}{c}
\Pi \\
\Omega
\hline
\neg T\ell \\
T\ell
\end{array}
\quad
\begin{array}{c}
\bot
\end{array}
$$

where it is obvious, from the context, what the proof Ω is. Upon applying $\neg$-Reduction, we obtain

$$
\begin{array}{c}
\Omega \\
T\ell
\hline
\ell = \neg T\ell
\end{array}
\quad
\begin{array}{c}
\frac{T \neg T\ell \quad \Omega}{\bot}
\end{array}
$$

But note that Ω contains Π, which proves the same conclusion, $\neg T\ell$, as does the left immediate subproof in the last proof-display; and it uses the same set of premisses (namely, $\emptyset$). So the left immediate subproof in the last proof-display is prolix; it can be replaced by Π. That step of reduction (Avoiding Prolixity) produces as its result the construction

$$
\begin{array}{c}
\Pi \\
\Omega
\hline
\neg T\ell \\
T\ell
\end{array}
\quad
\begin{array}{c}
\bot
\end{array}
$$

which was the departure point for our attempted sequence of reductions! So we have discovered that the reduction sequence loops. We do not have, in (Λ),
a genuine proof of \( \bot \) from no assumptions. And this is the case despite the fact that we do have a normal-form proof of \( \neg T \ell \) and a normal-form disproof of \( \neg T \ell \). Such is the mystery — indeed, the unique distinguishing mark — of paradox. The global disproofs associated with them cannot be reduced to normal form.

For this reason also, the discovery of a genuine paradox occasions no need at all for rational belief-revision. One is obliged to revise one’s beliefs about the world only when a new belief that one decides to adopt is genuinely inconsistent with one’s present beliefs. With paradoxes, however, on the diagnosis advanced here, there is no genuine inconsistency to be disposed of.

### 4.3.3 A fly in the classical ointment

I remarked above that the anti-realist’s diagnosis of paradoxicality is one that is afforded by her choice of intuitionistic or constructive logic as the appropriate logic in which to regiment the reasoning involved, and that the classical logician who avails himself of strictly classical rules of negation might miss the classificatory insight vouchsafed to the anti-realist on account of her constructivism. It is time now to show how this is so, by giving a strictly classical global disproof that regiments a stretch of strictly classical informal reasoning associated with the Liar. The classical logician, in brief, has to confront the following derivation of \( \bot \) from no assumptions at all.

(Recall that all parties are agreed that one may use as an axiom the identity \( ‘\ell = '\neg T \ell’.\))

Recall

\[
\Sigma : \quad \begin{array}{c}
\neg T \ell \\
\hline
T \ell
\end{array}
\]

With a single step of classical reductio, the classical logician concludes that \( \ell \) is true:

\[
\begin{array}{c}
\neg T \ell \\
\hline
(1)
\hline
\Sigma
\end{array}
\]

\[
\begin{array}{c}
\Sigma \\
\hline
CR \quad \bot \\
\hline
(1)
\end{array}
\]

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Now recall the following immediate subproof of our earlier proof \( \Pi \):

\[
\begin{align*}
T \ell & \quad \ell = \neg \neg T \ell \\
\neg T \ell & \quad T \ell \\
\hline
\bot
\end{align*}
\]

If we graft copies of the foregoing classical proof of \( T \ell \) onto the two premiss-occurrences of \( T \ell \) in this last proof, we obtain

\[
\begin{align*}
\frac{\neg T \ell}{\text{(1)}} & \quad \frac{T \rightarrow \neg T \ell}{\text{(1)}} & \quad \ell = \neg \neg T \ell \\
\frac{T \ell}{\text{(1)}} & \quad \frac{\neg T \ell}{\text{(1)}} & \quad \frac{T \rightarrow \neg T \ell}{\text{(1)}} & \quad \ell = \neg \neg T \ell \\
\frac{\neg T \ell}{\neg T} & \quad \frac{T \rightarrow \neg T \ell}{T \ell} \quad \frac{T \ell}{\bot}
\end{align*}
\]

Now here’s the classical rub: this proof appears to be in normal form. The use of classical reductio has masked the real defect that lies at the heart of paradoxical reasoning (according to my account)—the abnormality that makes itself evident only when one hews to a constructivist line, as the anti-realist is committed to doing.

One can imagine the realist, or classical logician, objecting at this point:

But how can you be so sure that there are not any strictly classical paradoxes? Perhaps there are paradoxes whose associated proofs of \( \bot \) have to make use of strictly classical rules, and which cannot be emulated by the constructivist?

This time, the objection is not well-taken. It is beyond the scope of this study to give detailed reasons for dismissing the possibility described by the objector. I make the following methodological conjecture:

Paradoxes are never strictly classical. The kind of conceptual trouble that a paradox reveals will afflict the intuitionist just as seriously as it does the classicist. Therefore, attempted solutions to the paradoxes, if they are to be genuine solutions, must be
available to the intuitionist. Nothing about an attempted solution to a paradox should imply that the trouble it reveals lies with strictly classical moves of reasoning.

This conjecture is argued for at much greater length in Tennant 2013b.

Rigorous analysis of the deductive reasoning behind all the known paradoxes confirms that it can be constructivized. Thus far I have not been made aware of any counterexample to the conjecture—not even for ‘self-referential’ systems containing arithmetic. The reason why the semantic paradoxes are such a threat is that they arise for the constructive relevantist (the character more pithily called the core logician) whenever they arise for the classicist. Using strictly classical rules in order to display the devastation they wreak obscures just how devastating the paradoxes really are.

To the stubborn critic who remains unconvinced by this (perforce brief) reply to the last objection raised, I counter with a challenge, in defence of my conjecture above: Give an example of a semantic paradox whose associated deductive reasoning cannot be constructivized. At present, all the known paradoxes, and their constructive regimentations, indicate that an example of this kind would be remarkable indeed, and would occasion much re-thinking.

4.3.4 An objection to my proof-theoretic diagnosis

At this point a critic might object:

That’s all well and good. You have illustrated your diagnosis of paradox in connection with The Liar. Let us grant that you can do the same with the other familiar paradoxes of this kind—Grelling’s paradox, the Postcard Paradox, Curry’s Paradox, Tarski’s Quotational Paradox, and the like. Still: you will have collected only a rag-bag of paradoxes. You take, for each of them, a particular global proof of absurdity ‘associated’ with it. And you show that these particular proofs appear not to be normalizable. But we want general results! We want you to show that no paradox generates any constructive, relevant, normal-form proof (i.e., core proof) proof of absurdity. The non-normalizability of any particular global proof of absurdity (associated with a given paradox) does not guarantee that it has no associated global core proof of absurdity at all.

The objection is well-taken. And it is not at all embarrassing to concede that the general demand, at this stage, cannot (alas) be met. For we are still
looking. We are still at the stage of data-gathering. We are, however, discovering patterns in the data—patterns that are arresting, that have not been extensively investigated since first being noted, that appear not to be discernible in areas of ordinary reasoning or disputation, yet are common to the whole cluster of what philosophers and logicians call the logico-semantic paradoxes. We are, if you will, at the stage of a Kepler looking at planetary motions. We still await our Newton. Kepler had to draw attention to the conics in the heavens before a satisfactory explanation of them could be given. It is not unreasonable to present data, and the patterns that appear to be emerging from them, even though we are not yet in possession of a deeper theory that would explain them.

That much having been conceded, however, and that much of a mitigating methodological perspective having been provided, there is some stiffer resistance that the anti-realist can offer this sceptical critic. As the reader will be able to verify independently, the proof-theoretic phenomena to which I am seeking to draw attention are very robust. I had no particular ideological axe to grind when first discovering them. Rather, I analysed the common (constructive!) informal global derivations of apparent absurdity that can be found in any textbook on the paradoxes—passages of quite rigorous, but informal reasoning, characteristically ending with the triumphant claim ‘Contradiction!’. I sought simply to formalize, or regiment, these passages of reasoning as (intuitionistic) natural deductions. This can always be done very directly. And what emerged was the characteristic pattern reported here . . . in one example of paradox after another. This can hardly be an accident. It is certainly not the result of ‘massaging the data’. These are the data! Look!

Although I have been commending to the reader advice analogous to that which Wittgenstein gave to those who worried about necessary and sufficient conditions for an activity to be called a game, I must now confess that I do not intend to maintain a kind of Wittgensteinian ‘dogmatic defeatism’ about the possibility of a systematic account of the semantic paradoxes. In fact, far from it—I believe that the looking carried out thus far does not support any Wittgensteinian scepticism about the prospects for a rigorous and precise statement of necessary and sufficient conditions for paradoxicality. When one ventures into some terrain expecting diversity and idiosyncracy and irregularity, but becomes aware, in due course, of some striking regularity at the heart of a significant sample of specimens, one is inclined to conjecture that one has lit on some deep mark underlying the phenomena. And this is what should strike any theorist who first takes the time, and makes the effort, to look closely at the reasoning involved in the various
semantic paradoxes. Their would-be global disproofs fail to normalize.

Given the simplicity of the Liar Paradox, there is no other reduction sequence to be considered than the one that we found to loop. Our regimentation of the constructive reasoning involved in the Liar Paradox really does not admit of any regimented alternatives. The would-be constructive ‘proof of ⊥’ (the proof called (Λ), which would have been a proof of ⊥ from the empty set) forced itself upon us as the only regimented possibility. Anyone who doubts this can be invited to provide an alternative regimentation; and, if they succeed in doing so, to reduce it to normal form. It is reasonable to claim, with moral certainty, that they will not succeed. The anti-realist can claim to have uncovered here ‘what is really going on’ in the reasoning associated with The Liar.

Likewise with every other standard paradox. The constructive deductive reasoning associated with it, which supposedly constitutes a reductio, admits of just one, constructive and relevant, formalization. As one searches for the reductio-shaped intuitionistic natural deduction, there is but one path to follow within the search space.

This is unlike what happens with automated deduction in a general setting, where proof-search strategies are non-deterministic. The search space affords, in general, more than one alternative route to success. This is witnessed by the structurally distinct proofs that can be found upon termination of successful search, at various leaf-nodes of different branches of the search-tree within the search space. With the standard ‘non-empirical’ paradoxes, it would appear that there are no choices to be made—no multiple possibilities afforded by ‘branching points’ in the search space.

This needs to be borne in mind when considering the imaginary critic’s demand ‘We want you to show that no paradox generates any core global proof of absurdity.’ If, for each of the extant paradoxes, there is but one global ‘proof’ of absurdity to be found, given the syntactic form of the paradoxical sentence(s), and it turns out that the (dis)proof in question cannot be reduced to normal form, then this demand is less a criticism than a challenge. It identifies an interesting and urgent research project in

\[\footnote{I am confining attention to the ‘pure’ paradoxes here, while well aware of the phenomenon to which Kripke drew attention, of sets of sentences of a semantically closed language that turn out to be paradoxical only because of the way the empirical facts happen to be. Suppose Nixon says ‘Everything Dean says is false’, and Dean, as it happens, makes only one assertion: ‘Snow is white.’ No paradox arises. Dean’s utterance is true, and Nixon’s is false. But if Nixon says only ‘Everything Dean says is false’, and Dean’s sole assertion is ‘Everything Nixon says is true’, then we have paradox. We cannot stably assign truth-values to either of their assertions.} \]
the logic of semantically closed languages, a project whose technical results would have potentially important philosophical implications.

As for the critic’s next observation — ‘The non-normalizability of any particular global proof of absurdity (associated with a given paradox) does not guarantee that it has no associated normal-form global proof of absurdity at all’ — we can demur here as well, for the same reasons. If there is but one proof of absurdity, and it defies reduction to normal form, then there is indeed no associated normal-form proof of absurdity at all.

4.3.5 Revenge

A prominent recent theme in the logical literature on paradox has been that of revenge paradoxes (see, for example, Beall 2008) which beset even those formal theories (such as that of Kripke (1975)) that are able to characterize paradoxes in semantically closed languages by means of semantic methods that employ transfinite sequences of evaluations and so-called ‘fixed points’. One method of evaluation employed by Kripke gives rise to the notion of a ‘gap’ (the classification intended for paradoxes such as the familiar Liar). But then a revenge paradox arises as soon as one considers ‘This sentence is either a gap or false’. Suffice it to say here that the reader can rest assured that revenge paradoxes such as these evince the proof-theoretic form just identified. They display the characteristic symptom by means of which I propose to diagnose paradoxicality — the would-be ‘global disproof’ associated with the revenge paradox cannot be brought into normal form. (As in the case of the Liar and other standard paradoxes, the reduction procedure loops.) Interestingly, Tennant 1995 did not address revenge paradoxes; but would have ventured the conjecture about non-normalizability as applying to them. So the more recent discovery that revenge paradoxes evince the same logical pattern can be seen as confirmation of my conjecture.

Since those global disproofs cannot be brought into normal form, they do not reveal a genuine problem with what (it is purported) has been ‘disproved’! One would have a genuine problem only if the global ‘proof of ⊥’ were in normal form.

Now, it is true — as Scharp has observed — that, even though the Liar sentence ℓ itself gives rise to no global disproof, in normal form, of any of our theoretical principles, we nevertheless find ourselves in the following situation, as far as the Liar is concerned. We can give a proof of The Liar in normal form (Lemma 3). We can also give a disproof of The Liar in normal form (Lemma 4). But — and here’s the rub — we cannot put these two constructions together to obtain, by imagined transitivity, a global proof of
absurdity! This is because, when the two constructions are put together — in what their assembler might think is a quick ride to absurdity — the new construction (Λ) displayed above simply will not normalize!

We are now in a position also to answer the objection anticipated earlier, about the feared interaction of Lemmas 1 and 2. It is an instructive exercise to take the proof of Lemma 1, and replace \( \theta \) throughout by \( T \{\phi\} \). Call the resulting proof \( \Pi \).

\[
\begin{align*}
\Pi : & \\
\quad & \vdash \neg(T^\wedge \phi \lor \neg T^\wedge \phi) \\
\quad & \vdash T^\wedge \phi \lor \neg T^\wedge \phi \\
\Rightarrow & \\
\quad & \vdash \neg(T^\wedge \phi \lor \neg T^\wedge \phi) \\
\quad & \vdash T^\wedge \phi \lor \neg T^\wedge \phi \\
\quad & \bot
\end{align*}
\]

Then append copies of the proof of Lemma 2 to the undischarged assumption occurrences of \( \neg(T^\wedge \phi \lor \neg T^\wedge \phi) \) in \( \Pi \). It will be found that the resulting would-be proof of \( \bot \) does not normalize. The reduction sequence enters a loop.

So it might appear that the anti-realist can actually sail very close to the wind, asserting \( \neg D\ell \), without pain of genuine inconsistency. It might appear that she can actually enjoy a way of expressing her view that the Liar (like other paradoxical sentences of a semantically closed language) is an exception to the Principle of Bivalence.

But any enthusiasm along these lines would be premature. Just as the anti-realist must refrain from asserting the Liar, despite having a normal-form proof of its truth (because there is also a normal-form proof of its falsity), and must refrain from rejecting the Liar, despite having a normal-form proof of its falsity (because there is also a normal-form proof of its truth), so too, now, she must refrain from rejecting the determinacy of the Liar (despite having a normal-form proof of \( \neg D\ell \)), and refrain from asserting its determinacy. This is because the normal-form proof of the truth of the Liar yields a normal-form proof of its determinacy by a single application of \( \lor\)-I; and, in exactly the same way, the normal-form proof of the falsity of the Liar yields a normal-form proof of its determinacy.

If one takes either one of these normal-form proofs of the Liar’s determinacy, along with the normal-form disproof of the Liar’s determinacy furnished in Lemma 2, and fancies that one can thereby derive \( \bot \) by a final step of \( \neg\)-E, one will be thwarted in the now-familiar way. The resulting
would-be global disproof is not in normal form, and cannot be converted into normal form. The reduction procedure loops.

So anti-realist scruples about assigning either of the truth-values to the Liar have to be accompanied by similarly grounded scruples about claiming (or denying) that the Liar is determinate. The anti-realist must not try to say, for herself, that the Liar is a wonderful example supporting her ‘indeterminacy’ view. She must leave it to others to make that point on her behalf.

There is nothing new in this circumstance, as far as the anti-realist (or intuitionist in mathematics, say) is concerned. One will frequently hear remarks, from classical mathematicians, such as ‘For the intuitionist, Goldbach’s Conjecture might have no truth value’. This is the sort of thing that has to be left to the classicist to say. The intuitionist cannot characterize her own position in that way. She cannot say, in a cooperative tone of voice ‘Yes, indeed, it could be the case that Goldbach’s Conjecture is neither true nor false.’ For that would be to contradict herself. To attribute falsity, for the anti-realist (and for the intuitionist in mathematics) is to deny truth:

\[ \varphi \text{ is false} \iff \text{it is not the case that } \varphi \text{ is true.} \]

So, in denying that Goldbach’s Conjecture is true, the anti-realist is ipso facto claiming that it is false; hence she cannot claim that it is not false either. This was the lesson of Lemma 1. And the proof of absurdity here is in normal form.

4.3.6 Some (more finely)-tuned proof-theoretical ideas

Complementing the proof-theoretic account of paradox is the proof-theoretically motivated account I have offered of how to relevantize proofs (including proofs of absurdity, i.e. disproofs) by taking the process of normalization one step further, as it were, and eliminating all essential uses of the absurdity rule (also known as the rule of \textit{ex falso quodlibet}). Slight (and independently motivatable) adjustments are needed both to the rule of conditional proof and to the rule of proof by cases; details can be found in chapter ten of Tennant 1997, ‘On finding the right logic’, and also in Tennant 1992 and Tennant 2002. These adjustments are entirely natural, and in keeping with the usual left-to-right reading of the truth-tables for the conditional and disjunction. The basic idea is to tighten up the otherwise lax formulations of the natural-deduction rules for introducing or eliminating the standard logical operators. We disallow applications of discharge-rules that would call
for ‘vacuous discharge’. This phrase adverts to subordinate proofs (for the application of a given rule) in which one has not actually made any use of the assumption(s) that one is entitled to discharge by an application of the rule in question.

Two basic points need to be made about this method of relevantizing proofs. First, it coheres with widely shared intuitions about extremal cases of logical consequence (such as Lewis’s First Paradox $A, \neg A : B$). Secondly, it vouchsafes completeness, with the further welcome possibility of occasional epistemic gain. If the sequent $\Delta : \varphi$ is valid, then there is a relevantized proof either of $\varphi$ or of $\bot$ from (some subset of) $\Delta$. In fact, one can go so far as to define proofs so narrowly that (i) they must be in normal form, and (ii) normality requires that all major premisses of eliminations ‘stand proud’, with no proof-work above them. I call such proofs (when they contain no applications of classical negation rules) core proofs. As already noted, one can establish the important property of transitivity of core proof, with possible epistemic gain: if one has two core proofs, one of $\Delta : \varphi$ and the other of $\Gamma, \varphi : \psi$, then one can effectively determine from them a core proof either of $\Theta : \psi$ or of $\Theta : \bot$, for some subset $\Theta$ of $\Delta \cup \Gamma$.8

The proof-theoretically minded anti-realist can take core proof as the formal explication of the notion of warrant, or of canonical proof (and the notion of core disproof as the formal explication of the correlative notion of canonical disproof). Intriguingly, every logical or semantic paradox can be generated by using only the intuitionistic and ‘relevantized’ rules of core proof. I repeat my earlier methodological conjecture: Strictly classical moves of inference are never needed in order to ‘establish’ any logical or semantic paradox. Intuitionistic (and relevant) rules of reasoning always suffice for this purpose.

Paradoxicality, then, is generated at a very deep logical level—within

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7There is a single exception: one half of the rule of $\rightarrow$-Introduction. This rule has two halves. One of them is the usual rule of conditional proof, which permits one to discharge assumption-occurrences of the antecedent, but which does not oblige one to have used the antecedent as an assumption in the subordinate proof. (That is the exception mentioned.) The other half of the rule allows one to infer a conditional from a disproof of its antecedent (thereby discharging assumption-occurrences of the same); and it also obliges one to have used the antecedent as an assumption for the subordinate disproof in question.

8See Tennant 2012b. This result is an unexpectedly powerful unifier: it affords quick proofs, as corollaries, of the Gödel-Glivenko-Gentzen Theorem (if $\Delta \vdash_C \varphi$, then $\neg\neg\Delta \vdash_I \neg\neg\varphi$); and of the result that any strictly classical propositional proof can be converted into a form in which the rule of classical reductio ad absurdum is applied only once, and indeed as the proof’s final step. A paper on the unusual proofs of these corollaries afforded by the result on cut for core proofs is in preparation.
core logic, innocent of the strictly classical rules of inference. That fact speaks to the ineradicable peculiarities of semantically closed languages. Core logic itself cannot be reformed; none of its principles can be jettisoned. For an argument to the effect that all of core logic is required for the reasoning that is essential to any rational process of belief-revision, see Tennant 2012a.

We are therefore in need of a bold and radical way to confront, and accommodate, the logical and semantic paradoxes. Given the harmonious and tightly formulated rules for the logical operators in core logic, and the absolutely obvious introduction and elimination rules for the truth predicate, the problem must lie in how, with certain sentences involving both reference to sentences and attributions of truth (and/or falsity), in certain interpretive contexts (involving sometimes empirical facts about who uttered what sentences), the interplay among ‘local’ rules that are perfectly in order reaches the occasional ‘global’ impasse. No determinate truth-value succeeds in being conferred; and we just have to recognize that fact, and learn to live with it.

What is needed, then, is a criterion by means of which we can tell that we are in just such a situation — a situation where ‘things do not work out right’ for the determination of a (unique) truth value. The anti-realist is already primed for such outcomes by having reconciled herself to the possibility of declarative sentences, with determinate meanings, for which we can work out no determinate truth value. With the paradoxes, the current proposal is that we are dealing with an interesting new variety of such sentences, whose failure to have a truth value is owing, at least in part, to certain workings of our language. The criterion of non-terminating reduction sequences is an anti-realist attempt to delineate which sentences fail, in this particular and peculiar way, to have a truth value.

4.4 This marriage was made in heaven

What are we to make of this situation? I commend to the reader a happily quietist, anti-realist, reaction. Sit back, fold your arms, and shrug, perhaps with a quizzically raised eyebrow. What we are presented with is a sentence — the Liar, say — for which the grounds for its assertion and the grounds for its rejection appear to be equally matched. So who is going to rush to enter a verdict? The verdict cannot, remember, be ‘Aha! There’s an inconsistency here’ — for that verdict, as we have just seen, is ruled out for want of an appropriate global disproof in normal form. So I commend, in such a case, an attitude of resigned indifference. The anti-realist, remem-
ber, does not have to take a stand, with regard to any particular sentence, on its having a truth-value, until such time as she is given a (method to determine a) normal-form proof, or a (method to determine a) normal-form disproof, and some principled reason to believe that no undercutting example of the other kind of construction (disproof, given proof — or vice versa) is in the offing.

There are no genuine proofs of absurdity eligible to discombobulate the anti-realist theorist of paradox who is willing to push further the proof-theoretic ideals of harmony and normalizability in an attempt to find the kind of diagnosis of paradox that has the happy result that no treatment is needed. Readers all too often simply acquiesce in the claims made by theorists of paradox, to the effect that such-and-such type of self-referring sentence ‘generates paradox’ (whether immediately and a priori, or by courtesy of — or conspiracy with — certain untoward constellations of empirical facts). Or, rather, readers acquiesce in the assumption — which I am concerned here to expose and reject — that when paradox has been generated, inconsistency has thereby been established. This is emphatically not the case.

Our proof-theoretic diagnosis of paradoxicality tells us what paradoxicality consists in — to wit, the non-normalizability of a would-be global proof of absurdity. Such a global disproof is supposed to precipitate crisis, the way the discovery of any contradiction in one’s system of belief is supposed to precipitate crisis. But cooler, proof-theoretically informed, heads should be allowed to prevail. They can point out that actually, on closer and more careful examination, there is no crisis. This is because there is no inconsistency. And this in turn is because there is no normal-form proof of absurdity from anything to which we had reason to commit ourselves, and to which we would wish to remain committed.

The proof-theoretic criterion of paradoxicality has the advantage that it is more immediate, concrete and intuitive than the alternative, semantical, characterizations of paradoxicality that we variously owe to Woodruff and Martin, Kripke, Herzberger, Gupta and Belnap, and others. The proof-theoretic criterion gives point, also, to the anti-realist’s demurral over Bivalence. When the scales of evidence and inference are exactly balanced — as they are, say, with the Liar’s truth and the Liar’s falsity — then there is no need to enter the lists to lay claim to either truth value. One can remain disengaged and unruffled, precisely because one has discovered a logical or semantic paradox.
4.5 The revenge problem revisited

One can imagine an objector insisting that my ‘solution’ to the problem of paradox is given in a language that transcends, in expressive power, the language for which the solution is offered. My diagnosis of paradox, so the objection goes, brings to bear various proof-theoretical resources whose net effect is to enable us to say, of a given paradoxical sentence (or set of sentences) ‘This is paradoxical’. But that is to open the door to the revenge paradox ‘This sentence is false or paradoxical’. If we are to allow the latter into the language for which my solution is supposedly fashioned, then we shall have to diagnose it as paradoxical. But that would be to establish its truth. Shouldn’t one then assert it? — and thereby contradict oneself?

The objector here is overlooking, of course, the similarity of the revenge paradox to the Liar. Having ‘established’ its truth, it is quick work also to ‘establish’ its falsity (just as it was when the revenge paradox took the form ‘This sentence is either false or a gap’). And a little further work will reveal that the ‘global disproof’ one might think one could obtain by adjoining the proof of its truth with the proof of its falsity cannot be normalized. So the sentence ‘This sentence is false or paradoxical’ is indeed a paradox, in my proposed sense of paradox.

I make this point, however, not from within the anti-realist’s shoes, but from a metaphorical sideline. The anti-realist herself would, by contrast, be well-advised to refrain from saying of the revenge paradox that it is paradoxical. The anti-realist should refrain from asserting the revenge paradox; refrain from denying it; and refrain from classifying it. The anti-realist can, however, show that it is paradoxical by exhibiting the non-terminating reduction sequence on which one will embark if one tries to normalize the would-be ‘global proof’ of absurdity referred to earlier. That can lead her interlocutor to understand why she refrains from asserting it, or denying it, or even classifying it. But do not expect her to say that these are her reasons why. Whereof one cannot speak, one must pass over in silence.

4.6 Saying No to dialetheism

One can imagine another objector insisting that, given my concession that there is a normal-form proof of ‘ℓ is true’ and a normal-form disproof of the same, the proof-theoretically minded anti-realist should be ready and willing both to assert the claim ‘ℓ is true’ and to reject it. In other words, the anti-realist should become a dialetheist.

I do not regard this as a sufficiently motivated thing to do, once the full
epistemic situation is understood. (For a more sustained critique of dialethe-
ism from an anti-realist standpoint, see Tennant 2004.) This is because it
would be incorrect to represent the anti-realist as both warrantedly asserting
that $p$ and warrantedly asserting that $\neg p$, in cases where a paradox is in-
volved. Certainly, upon first discovering, say, the normal-form proof of ‘$\ell$ is
true’, one might be tempted to assert it. But preparedness to assert rests not
only on having an apparent warrant, but also on being convinced that there
is no defeater for that apparent warrant. In mathematics, for example —
given the normalizability of proof — the mere discovery of a proof suffices.
This is because, in the background, we have the overarching assumption that
mathematics is consistent — given a normal-form proof of $\varphi$, there cannot be
any normal-form disproof of $\varphi$. If there were, the two constructions could be
combined to reveal an inconsistency in one’s mathematical foundations —
and the resulting construction would be normalizable!

But that is not how we are allowed to proceed in the regions of semanti-
cally closed languages, regions in which, as we know, there are many exotic
threats posed to careless travelers. Having in hand a normal-form proof
of ‘$\ell$ is not true’ affords no guarantee at all that one will not be able to
discover — as indeed one subsequently does, in short reflective measure — a
normal-form proof of ‘$\ell$ is true’. See an analogous point raised in connection
with Milne’s sentence $M$ below.

5 On other ungrounded sentences

The proof-theoretic criterion of paradoxicality that is being commended here
is obviously related, in some way that merits further investigation, to the ‘un-
grounded’ character, on the semantical accounts, of paradoxical sentences.
What I say, on anti-realist grounds, about the Liar sentence holds equally
well for the (non-paradoxical, but) ungrounded sentence called the Truth-
teller:

$$(\tau) \quad \tau \text{ is true.}$$

There is no definitive reason for regarding it as true, and there is no definitive
reason for regarding it as false. So, the anti-realist will not be moved to
regard it as either.

There are other ungrounded sentences, however, which, because they are
ungrounded, are deprived (on the semantical accounts) of the status of being
true, but which we are strongly tempted to advance as general principles that
must be true. In closing this discussion, I shall briefly touch on two of these.

1. A principle like

\begin{quote}
A biconditional is true if and only if its immediate constituents
are either both true or both false
\end{quote}

is one that most theorists would like to be able to regard as true. Yet it
is ungrounded in semantically closed languages. How, then, can one claim
for it the truth value True? (Bear in mind that the principle is itself a
biconditional!)

I have a suggestion concerning general principles like this one: although
the non-normalizability of a purported global proof of absurdity might be
a reliable sign that one is dealing with a paradox, the absence of any well-
founded sequence of evaluations leading to a stable assignment of True to a
sentence $\varphi$ is not to be taken as a sign that there is no normal-form proof
of $\varphi$. Indeed, I would furnish a set of inferential rules for the theory of
truth (see Tennant 1987a and Tennant 1987b) telling one the grounds for
truth-attributions to logically compound sentences, which would allow one
to construct a normal-form proof of the foregoing biconditional. There is no
indication whatever, from the logical structure of the claim in question, and
of its normal-form proof, that there is any danger of a normal-form dis-
proof of the same lurking out there, waiting to be discovered. This is because the
truth-maker on offer does not rummage around in any relations of referen-
tiality (self- or otherwise).

2. The second example of a (semantically) ungrounded principle is

\begin{quote}
Every true sentence has a truth-maker.
\end{quote}

Semantic theorists of semantically closed languages will of course point out
that this sentence could never be evaluated as stably true. But it is an
obviously appealing principle, especially to the anti-realist. Must the anti-
realist give it up (for semantically closed languages), just because of these
alleged problems of grounding?

I think not. The principle is true by definition, since a true sentence, for
the anti-realist, is one for which there exists a truth-maker. So the principle
is an obvious substitution instance of the logically true schema ‘Every $F$ is
F’, which has a two-step truth-maker:

\[
\frac{\quad (1) \quad}{\quad F_a \quad (1) \quad} \quad \frac{\quad F_a \rightarrow F_a \quad}{\quad \forall x(F_x \rightarrow F_x) \quad}
\]

As soon as we realize that truth-makers are proof-like objects, and remind ourselves of the fundamental logical principle that any substitution instance of a logical theorem is logically true, it is hard to resist this easy positive argument for the truth of ‘Every true sentence has a truth-maker’.

Peter Milne (2005) offers the sentence

\[M: \text{‘This sentence has no truthmaker’},\]

as an alleged example of a true sentence without a truthmaker. In order to show that \(M\) is true, Milne provides a proof establishing \(M\); whereupon he concludes that \(M\) has no truthmaker, because ‘this is just what \(M\) says’ (p. 222). But Milne omits to remark that his proof of \(M\), being itself a truthmaker for \(M\), also falsifies \(M\). So the anti-realist who holds the position on paradox put forward here will remain unconvinced by Milne’s argument. That anti-realist will say that Milne’s sentence \(M\) is not a counterexample to the general claim, defended above, that every true sentence has a truthmaker.

If an objector finds my positive argument above for the truth of ‘Every true sentence has a truth-maker’ a little too insouciant, so be it. Just as Wittgenstein expressed his puzzlement over why anyone would criticize him for attacking an opponent’s theory at its weakest point, so too I would be puzzled at the suggestion that one should not defend a claim by advancing for it the strongest possible proof. Two logical steps, followed by a substitution, seem to me to fit the bill. Again, there does not appear to be any disproof lurking out there, given the fact, once again, that the truthmaker on offer does not rummage around in any relations of referentiality. If, however, a resourceful opponent can find a normal-form disproof of the truth-maker principle, then this particular anti-realist would be prepared to beat a hasty retreat, to contemplate yet another dropped stitch in the logical fabric of the world. There is nothing to be ashamed of when one is prepared to undertake whatever rational revision of one’s beliefs might be called for.  

\[\text{9} \]

\[\text{9I would like to thank Adam Podlaskowski and Kevin Scharp for their encouragement to write this piece.} \]
References


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