

Stochastic Differential Equations

SSES, Spring 2015

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Recall:

- we introduced the idea that many processes can be described using (partial) differential equations. $dX(t) = \dots dt$
- such equations do not capture **everything**.
- one solution is to add noise to the PDEs HOW ?

Today :- basic tools

- SDEs (processes varying in time only)

Random variables, random vectors, random sequences

$$(\Omega, \mathcal{B}, P)$$

$$X: \Omega \rightarrow \mathbb{R}$$

random variable

X_1, X_2 - random variables

$$X = [X_1, X_2]$$

$$X: \Omega \rightarrow \mathbb{R}^2$$

$$X = [X_1, \dots, X_k]$$

$$X: \Omega \rightarrow \mathbb{R}^k$$

random vector

$$X = [X_1, X_2, \dots, X_n, \dots]$$

$$X: \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$$

set of all sequences of real numbers

$$X = (X_n)_{n \geq 1}$$

$$w \rightarrow X(w) = [X_1(w), X_2(w), \dots, X_n(w), \dots]$$

$X = \boxed{\text{random sequence}}$

Stochastic processes

In general $X = (X_t)_{t \in \mathbb{T}}$ $\mathbb{T} = [0, T], [0, \infty)$

↳ a collection of random variables, indexed over $t \in [0, T]$

$X = (X_t)_{t \in \mathbb{T}} = \underline{\text{a stochastic process}}$ for each t $X_t: \Omega \rightarrow \mathbb{R}$

as a process, $X: \Omega \rightarrow \mathbb{R}^{[0, T]}$ → all functions $x: [0, T] \rightarrow \mathbb{R}$

random vector

$$X: \Omega \rightarrow \mathbb{R}^k$$

$$\omega \mapsto X(\omega) = x \in \mathbb{R}^k$$

$$x = (x_1, \dots, x_k)$$

stochastic process (random function)

$$X: \Omega \rightarrow \mathbb{R}^{[0, T]}$$

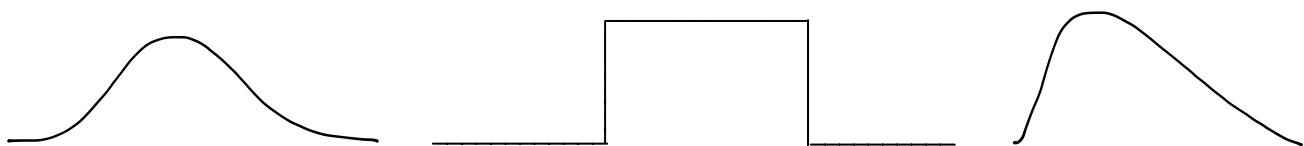
$$\omega \mapsto X(\omega) = x \in \mathbb{R}^{[0, T]}$$

$$x = \left(x(t) \right)_{t \in [0, T]}$$

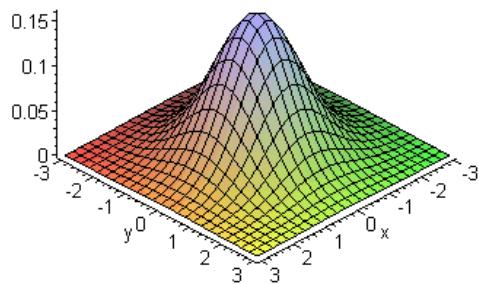
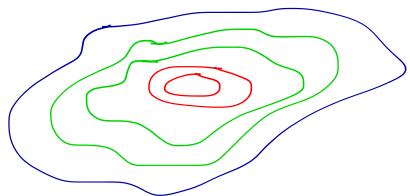
$$x: [0, T] \rightarrow \mathbb{R} \quad t \mapsto x(t)$$

Think distributions

random variable \rightarrow univariate distributions



random vectors \rightarrow multivariate distributions

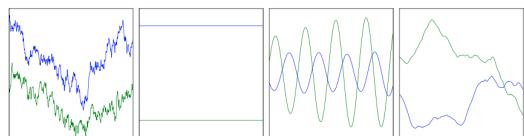
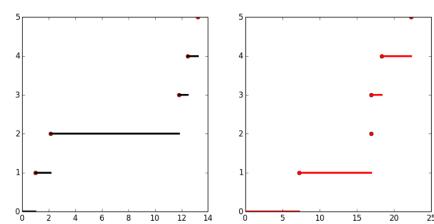
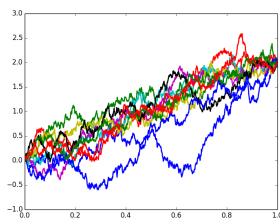
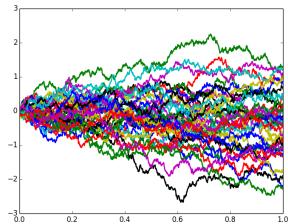


random functions \rightarrow infinite dim. distributions over spaces like

$$\mathbb{R}^N \quad \mathbb{R}^{[0,T]} \quad \mathbb{R}^{[0,\infty)} \quad \dots$$

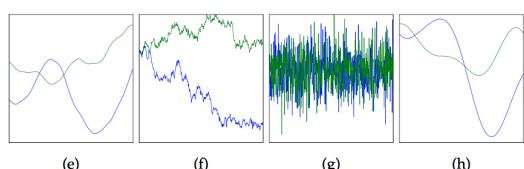
$$\mathbb{R}^{[0,T]} = \{x : x: [0,T] \rightarrow \mathbb{R}\}$$

these are difficult to visualize



Can we do statistical inference in $\mathbb{R}^{[0,T]}$
like we do in \mathbb{R}^2 ?

(define pdfs, find MLEs, Bayesian methods, ...)



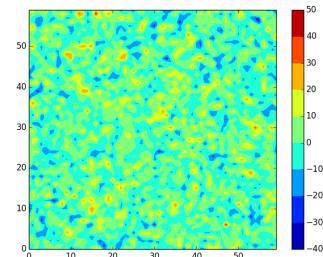
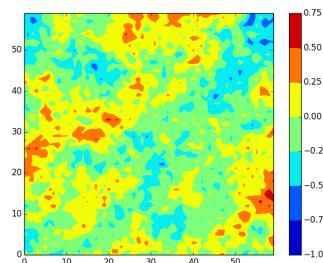
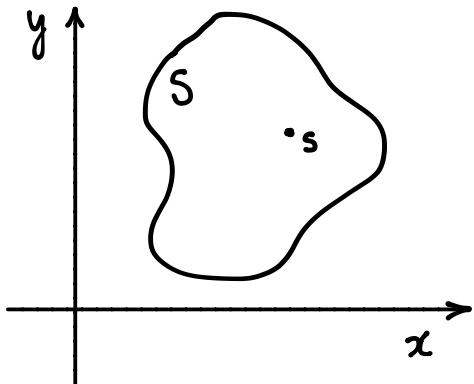
(curious? - never quit!)

Random fields

$(X_t) \quad t \in [0, T] \rightarrow$ random function $X: \Omega \rightarrow \mathbb{R}^{[0,T]}$

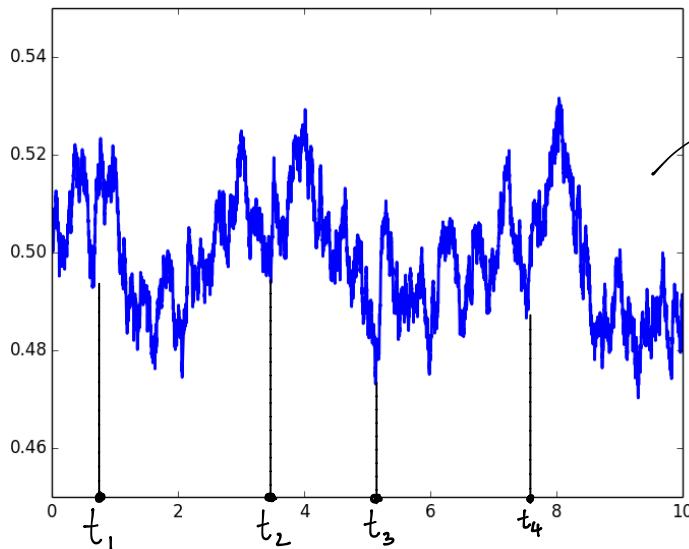
$(X_s) \quad s \in S \subset \mathbb{R}^2 \rightarrow$ random field $X: \Omega \rightarrow \mathbb{R}^S$

(now think distributions over surfaces defined on S)



Random functions and finite dimensional distributions

$(X_t, 0 \leq t \leq T)$ - random function (stochastic process) - a distribution $[0, T]$ on \mathbb{R}



X_{t_1} - r.v.

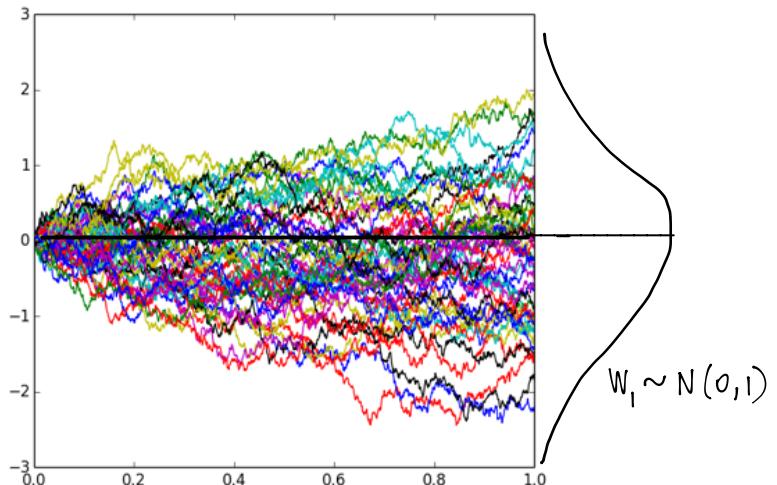
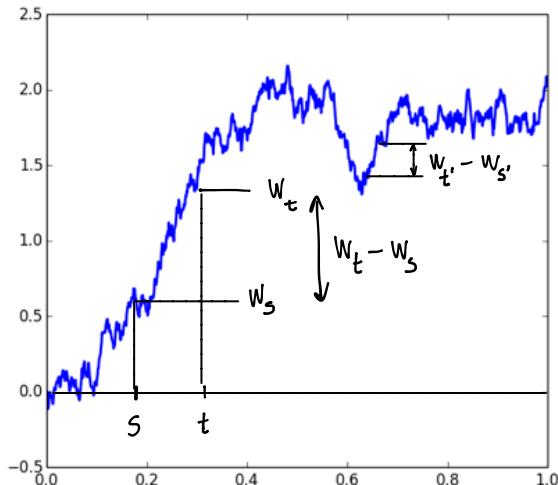
X_{t_2} - r.v.

X_{t_3}, X_{t_4} - r.v.

$[X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4}]$ - random vector what is its distribution?

Brownian Motion

$(W_t \quad 0 \leq t \leq T)$



$$W_0 = 0$$

independent increments $W_t - W_s \perp W_{t'} - W_s$

Gaussian increments $W_t - W_s \sim N(0, t-s)$

continuous sample paths

Stochastic integrals

(think W_t = price of a stock
at time t)

(W_t) - Brownian Motion process $t \in [0, T]$

for each t , W_t is a r.v. $W_t \sim N(0, t)$ $\text{cov}(W_s, W_t) = \min(s, t)$

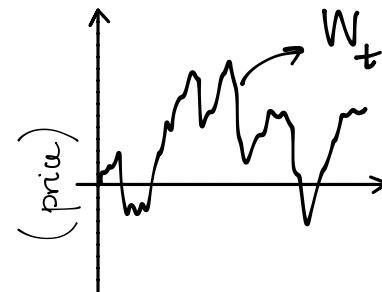
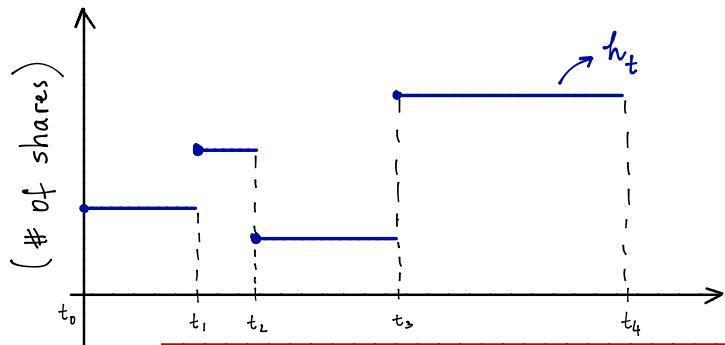
$\mathcal{F}_t = \sigma \left(W_s \mid 0 \leq s \leq t \right)$ = information available up to time t
(\mathcal{F} -field generated by W_0, \dots, W_t)

(h_t) $0 \leq t \leq T$ - a progressively measurable process

for any fixed t h_t = r.v.

given \mathcal{F}_t $(h_s \mid 0 \leq s \leq t)$ are known!

(think h_t = # of shares you own at time t)



Q: what is the profit up to time t ?

profit at $t \in [t_0, t_1)$

$$W_t \cdot h_{t_0} - W_{t_0} \cdot h_{t_0} = h_{t_0} (W_t - W_{t_0})$$

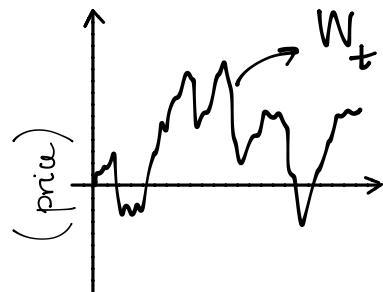
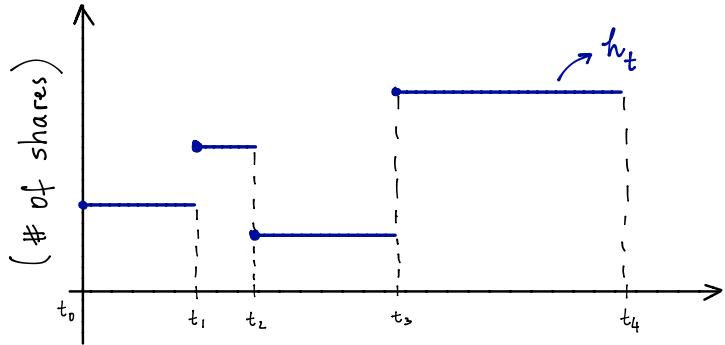
profit at $t \in [t_1, t_2)$

$$= \text{profit over } [t_0, t_1) + \text{profit over } [t_1, t)$$

$$= h_{t_0} (W_{t_1} - W_{t_0}) + h_{t_1} (W_t - W_{t_1})$$

In general: profit at $t \in [t_k, t_{k+1})$ is

$$h_{t_0} (W_{t_1} - W_{t_0}) + \dots + h_{t_k} (W_t - W_{t_k}) = \int_0^t h_s dW_s$$

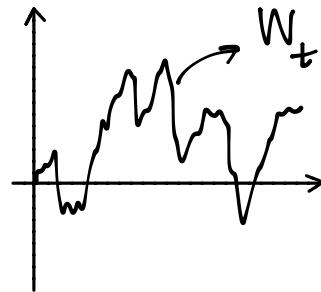
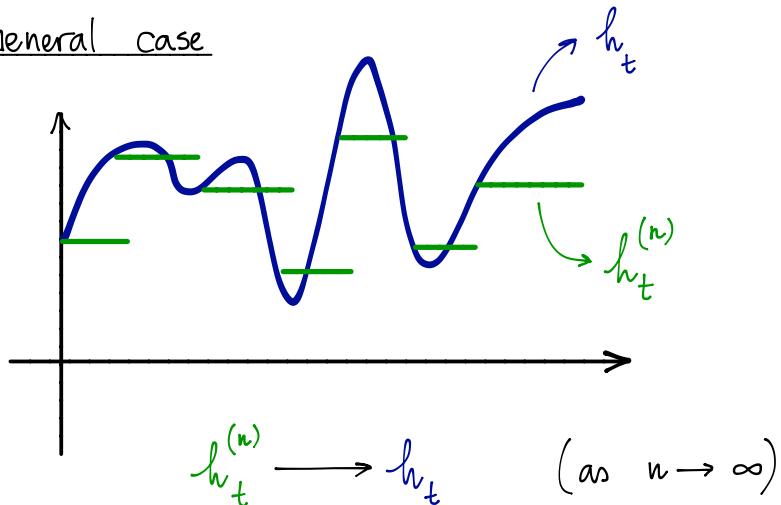


$$I_t(h) = \int_0^t h_s dW_s = h_{t_0}(W_{t_1} - W_{t_0}) + \dots + h_{t_k}(W_t - W_{t_k})$$

This is called the Ito integral of (h_t) wrt. (W_t)

- Q2 questions
- 1) find $E(I_t(h))$
 - 2) Find $\text{Var}(I_t(h))$
 - 3) Show that $I_t(h)$ is a martingale
 - 4) ...

General case



$$h_t^{(n)} \rightarrow h_t \quad (\text{as } n \rightarrow \infty)$$

$$\int_0^t h_s^{(n)} dW_s \rightarrow \int_0^t h_s dW_s \quad [\text{Itô integral of } (h_t) \text{ wrt } (W_t)]$$

$$I_t(h) = \int_0^t h_s dW_s$$

$$I_t(h) \sim N(0, \dots)$$

continuous sample paths

martingale, etc.

Stochastic (ordinary) Differential Equations (SDE)

- ① say $S(\cdot)$ and $\Gamma(\cdot)$ are smooth functions
- ② X_0 - some random variable
- ③ (W_t) - a Brownian motion process

A stochastic process (X_t) is an Itô process if

$$X_t = X_0 + \underbrace{\int_0^t S(X_s) ds}_{\text{r.v.}} + \underbrace{\int_0^t \Gamma(X_s) dW_s}_{\text{Itô integral}} \quad (\text{a.s.})$$

X_t is completely known given the inputs $X_0, (W_s, 0 \leq s \leq t)$

X_t is measurable wrt $\mathcal{F}(X_0, W_s, 0 \leq s \leq t)$

$\hat{\text{It\^o}}$ process : $X_t = X_0 + \int_0^t S(X_s) ds + \int_0^t \Gamma(X_s) dW_s$ (a.s.)

Informally, (X_t) satisfies the Stochastic Differential Equation (SDE)

$$dX_t = S(X_t)dt + \Gamma(X_t)dW_t \quad X_{t=0} = X_0$$

$S(\cdot)$ = drift function (coefficients)

$\Gamma(\cdot)$ = diffusion function

X_0 = initial condition (value)

$$dX_t = S(X_t)dt + \Gamma(X_t)dW_t \quad X_{t=0} = X_0$$

An SDE is :

- a way to describe a stochastic process
- a way to describe a distribution over a space of functions $\mathbb{R}^{[0,T]}$

Simple examples

$$dX_t = S(X_t)dt + \Gamma(X_t)dW_t \quad X_{t=0} = X_0$$

① $S(x) = 0 \quad \Gamma(x) = \Gamma$ (constant)

$$dX_t = \Gamma dW_t$$

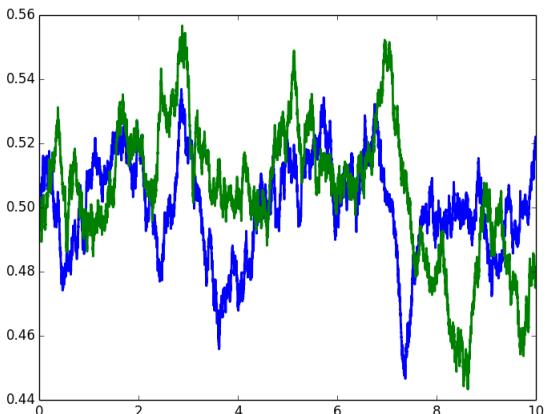
$$\begin{aligned} X_t &= X_0 + \int_0^t \Gamma dW_s = X_0 + \Gamma(W_t - W_0) \\ &= X_0 + \Gamma W_t \end{aligned}$$

(scaled and shifted BM)

② $S(x) = \theta_1 - \theta_2 x \quad \Gamma(x) = \Gamma$

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \Gamma dW_t$$

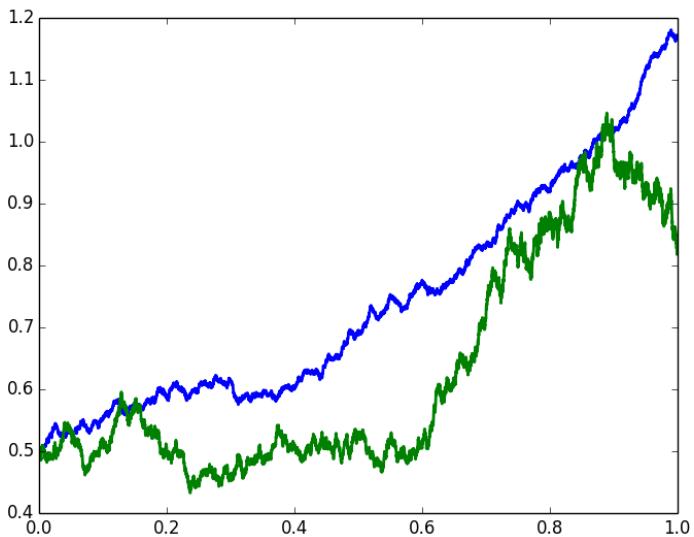
(Ornstein-Uhlenbeck process)



$$③ S(x) = \theta_1 x \quad \Gamma(x) = \theta_2 x$$

$$dX_t = \theta_1 X_t dt + \theta_2 X_t dW_t$$

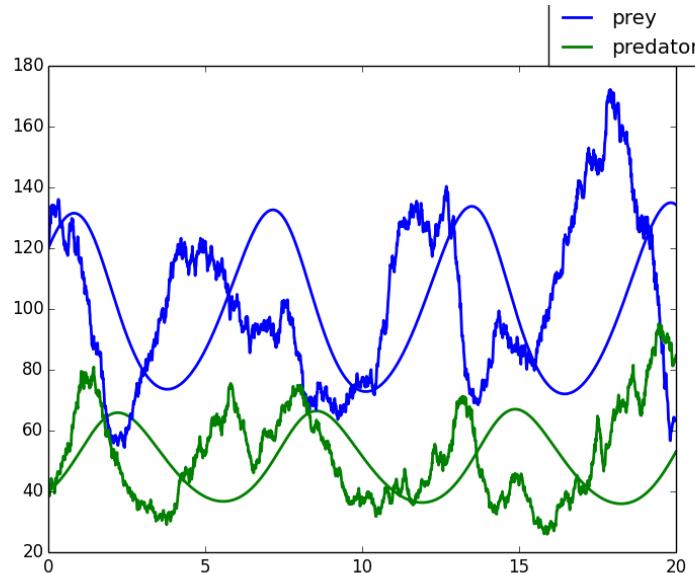
Geometric Brownian Motion



Old examples revisited

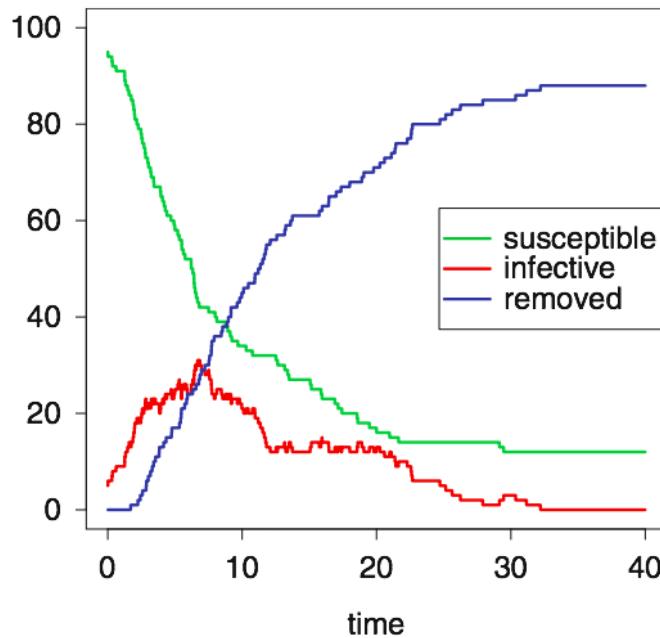
predator-prey model

$$\begin{cases} dX_t = X_t (a_{10} - a_{12} Y_t) dt + \sqrt{X_t (a_{10} + a_{12} Y_t)} dW_t^{(1)} \\ dY_t = Y_t (a_{21} X_t - a_{20}) dt + \sqrt{Y_t (a_{21} X_t + a_{20})} dW_t^{(2)} \end{cases}$$



SIR model

$$\begin{cases} dS_t = -\alpha S_t I_t dt + \frac{1}{\sqrt{N}} \sqrt{\alpha S_t I_t} dW_t^{(1)} \\ dI_t = (\alpha S_t I_t - \beta I_t) dt - \frac{1}{\sqrt{N}} \sqrt{\alpha S_t I_t} dW_t^{(1)} + \frac{1}{\sqrt{N}} \sqrt{\beta I_t} dW_t^{(2)} \end{cases}$$



$$dX_t = S(X_t)dt + \sigma(X_t)dW_t \quad X_{t=0} = X_0$$

How do we solve a SDE ?

$$X_t = (\dots)$$

\uparrow something involving X_0 and $(W_s \quad 0 \leq s \leq t)$

We don't !

- very few SDEs can be solved analytically !
 $(OU, GBM, CIR, \text{a few more...})$
- in all other cases people would use an approximation.

$$dX_t = S(X_t)dt + \mathcal{G}(X_t)dW_t \quad X_{t=0} = X_0$$

Euler approximation

- consider a very fine time grid $0 = t_0 < t_1 < \dots < t_N = T$

$$t_{i+1} - t_i = \Delta t$$

- $dX_t \approx X_{t+\Delta t} - X_t \quad dW_t \approx W_{t+\Delta t} - W_t \sim N(0, \Delta t)$

- construct the approximating process

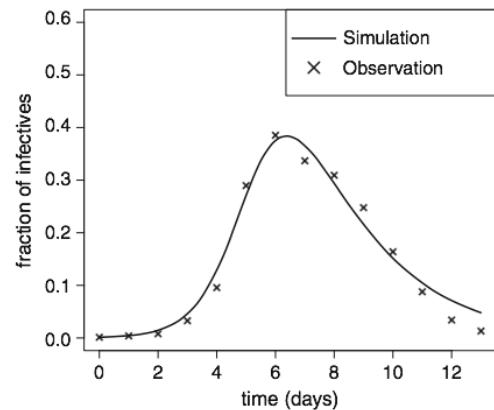
$$X(t_{i+1}) = X(t_i) + S(X(t_i)) \Delta t + \mathcal{G}(X(t_i)) \cdot N(0, \Delta t)$$

- this is bad if Δt is large.

Data vs. SDE models

Table 8.4 Daily number of boys confined to bed, reconstructed from the graphic displayed in the original publication (BMJ News and Notes 1978). The total number of boys visiting the school was $N = 763$. The fractions of infective boys are plotted in Fig. 8.7

Date	Number of boys confined to bed
21 January	1
22 January	3
23 January	6
24 January	25
25 January	73
26 January	221
27 January	294
28 January	257
29 January	236
30 January	189
31 January	125
1 February	67
2 February	26
3 February	10
4 February	3



Data: $[I(t_1)^{\text{obs}}, I(t_2)^{\text{obs}}, \dots, I(t_n)^{\text{obs}}]$

Goal: estimate α, β

$$\begin{cases} dS_t = -\alpha S_t I_t dt \\ dI_t = (\alpha S_t I_t - \beta I_t) dt \end{cases}$$

Data: $[I(t_1)^{\text{obs}}, I(t_2)^{\text{obs}}, \dots, I(t_n)^{\text{obs}}] \equiv I^{\text{obs}}$

Correct model

$$\begin{cases} dS_t = -\alpha S_t I_t dt + \frac{1}{\sqrt{N}} \sqrt{\alpha S_t I_t} dW_t^{(1)} \\ dI_t = (\alpha S_t I_t - \beta I_t) dt - \frac{1}{\sqrt{N}} \sqrt{\alpha S_t I_t} dW_t^{(1)} + \frac{1}{\sqrt{N}} \sqrt{\beta I_t} dW_t^{(2)} \end{cases}$$

Posterior density: $[\alpha, \beta | I^{\text{obs}}] \propto [\alpha, \beta] [I^{\text{obs}} | \alpha, \beta] \rightarrow \text{not available in closed form}$

Problem: the stochastic process $(I_t, S_t \mid 0 \leq t \leq T)$ induces a marginal distribution

$[I(t_1), \dots, I(t_n)]$ which is far too complicated to write down

Impossible situation:

- process under study is **continuous**, SDE seems the appropriate model

$$(X_t \quad 0 \leq t \leq T) \quad dX_t = \dots$$

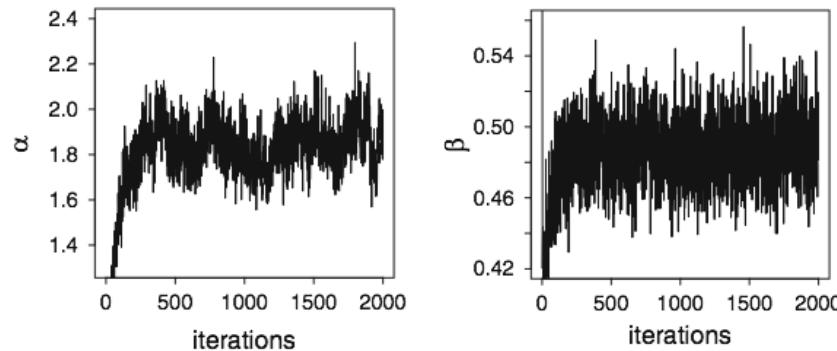
- data are **discrete**

$$[X(t_1), X(t_2), X(t_3), \dots, X(t_n)]$$

- likelihood not available !

Nobody panic !

- solutions exist
- they involve various approximations (or not !)



- interested ? take STAT 8540 ☺

Summary

- Brownian motion ($W_t \quad 0 \leq t \leq T$)
- Itô integral $\int_0^t h_s dW_s$
- stochastic differential equation $dX_t = S(X_t) dt + \Gamma(X_t) dW_t$
- Euler scheme \rightarrow approximating solutions
- (discrete) data vs. (continuous) process

Next: space-time modeling with SPDEs