5.7. Transfinite induction, comprehension, indiscernibles, infinity, $\Pi_1^0$ correctness.

We now fix $M\# = (D, <, \in, \text{NAT}, 0, 1, +, -, \cdot, \uparrow, \log, d_1, d_2, \ldots)$ as given by Lemma 5.6.18.

While working in $M\#$, we must be cautious.

a. The linear ordering $<$ may not be internally well ordered. In fact, there may not even be a $a <$ minimal element above the initial segment given by NAT.

b. We may not have extensionality.

Note that we have lost the internally second order nature of $M^*$ as we passed from $M^*$ to the present $M\#$ in section 5.6. The goal of this section is to recover this internally second order aspect, and gain internal well foundedness of $<$.

To avoid confusion, we use the three symbols $=, \equiv, \approx$. Here $=$ is the standard identity relation we have been using throughout the book.

DEFINITION 5.7.1. We use $\equiv$ for extensionality equality in the form

$$x \equiv y \iff (\forall z)(z \in x \iff z \in y).$$

DEFINITION 5.7.2. We use $\approx$ as a special symbol in certain contexts.

DEFINITION 5.7.3. We write $x \approx \emptyset$ if and only if $x$ has no elements.

We avoid using the notation $\{x_1, \ldots, x_k\}$ out of context, as there may be more than one set represented in this way.

DEFINITION 5.7.4. Let $k \geq 1$. We write $x \approx \{y_1, \ldots, y_k\}$ if and only if

$$(\forall z)(z \in x \iff (z = y_1 \lor \ldots \lor z = y_k)).$$

LEMMA 5.7.1. Let $k \geq 1$. For all $y_1, \ldots, y_k$ there exists $x \approx \{y_1, \ldots, y_k\}$. Here $x$ is unique up to $\equiv$.

Proof: Let $y = \max(y_1, \ldots, y_k)$. By Lemma 5.6.18 iv),
\[(\exists x) (\forall z) (z \in x \leftrightarrow (z \leq y \land (z = y_1 \lor \ldots \lor z = y_k))).\]

The last claim is obvious. QED

DEFINITION 5.7.5. We write \(x = <y, z>\) if and only if there exists \(a, b\) such that

i) \(x = \{a, b\}\);

ii) \(a = \{y\}\);

iii) \(b = \{y, z\}\).

LEMMA 5.7.2. If \(x = <y, z> \land w \in x\), then \(w = \{y\} \lor w = \{y, z\}\). If \(x = <y, z> \land x = <u, v>\), then \(y = u \land z = v\). For all \(y, z\), there exists \(x = <y, z>\).

Proof: For the first claim, let \(x, y, z, w\) be as given. Let \(a, b\) be such that \(x = \{a, b\}\), \(a = \{y\}\), \(b = \{y, z\}\). Then \(w = a \lor w = b\). Hence \(w = \{y\} \lor w = \{y, z\}\).

For the second claim, let \(x = <y, z>, x = <u, v>\). Let

\(x = \{a, b\}, a = \{y\}, b = \{y, z\}\)

\(x = \{c, d\}, c = \{u\}, d = \{u, v\}\).

Then

\((a = c \lor a = d) \land (b = c \lor b = d) \land (c = a \lor c = b) \land (d = a \lor d = b)\).

Since \(a = c \lor a = d\), we have \(y = u \lor (y = u = v)\). Hence \(y = u\).

We have \(b = \{y, z\}, d = \{y, v\}\). If \(b = d\) then \(z = v\). So we can assume \(b \neq d\). Hence \(b = c, d = a\). Therefore \(u = y = z, y = u = v\).

For the third claim, let \(y, z\). By Lemma 5.7.1, let \(a = \{y\}\) and \(b = \{y, z\}\). Let \(x = \{a, b\}\). Then \(x = <y, z>\). QED

DEFINITION 5.7.6. Let \(k \geq 2\). We inductively define \(x = <y_1, \ldots, y_k>\) as follows. \(x = <y_1, \ldots, y_{k+1}>\) if and only if

\((\exists z) (x = <z, y_3, \ldots, y_{k+1}> \land z = <y_1, y_2>)\).

In addition, we define \(x = <y>\) if and only if \(x = y\).

LEMMA 5.7.3. Let \(k \geq 1\). If \(x = <y_1, \ldots, y_k>\) and \(x = <z_1, \ldots, z_k>\), then \(y_1 = z_1 \land \ldots \land y_k = z_k\). For all \(y_1, \ldots, y_k\), there exists \(x\) such that \(x = <y_1, \ldots, y_k>\).
Proof: The first claim is by external induction on \( k \geq 2 \), the case \( k = 1 \) being trivial. The basis case \( k = 2 \) is by Lemma 5.7.2. Suppose this is true for a fixed \( k \geq 2 \). Let \( x = <y_1, \ldots, y_{k+1}> \), \( x = <z_1, \ldots, z_{k+1}> \). Let \( u, v \) be such that \( x = <u, y_3, \ldots, y_{k+1}>, x = <v, z_3, \ldots, z_{k+1}>, u = <y_1, y_2>, v = <z_1, z_2> \).

By induction hypothesis, \( u = v \land y_3 = z_3 \land \ldots \land y_{k+1} = z_{k+1} \).

By Lemma 5.7.2, since \( u = v \), we have \( y_1 = z_1 \land y_2 = z_2 \).

The second claim is also by external induction on \( k \geq 2 \), the case \( k = 1 \) being trivial. The basis case \( k = 2 \) is by Lemma 5.7.2. Suppose this is true for a fixed \( k \geq 2 \). Let \( y_1, \ldots, y_{k+2} \). By Lemma 5.7.2, let \( z = <y_1, y_2> \). By induction hypothesis, let \( x = <z, y_3, \ldots, y_{k+2}> \). Then \( x = <y_1, \ldots, y_{k+2}> \).

QED

DEFINITION 5.7.7. Let \( k \geq 1 \). We say that \( R \) is a \( k \)-ary relation if and only if \( (\forall x \in R)(\exists y_1, \ldots, y_k)(x = <y_1, \ldots, y_k>) \). If \( R \) is a \( k \)-ary relation then we define \( R(y_1, \ldots, y_k) \) if and only if

\[
(\exists x \in R)(x = <y_1, \ldots, y_k>).
\]

Note that if \( R \) is a \( k \)-ary relation with \( R(y_1, \ldots, y_k) \), then there may be more than one \( x \in R \) with \( x = <y_1, \ldots, y_k> \).

We use set abstraction notation with care.

DEFINITION 5.7.8. We write

\[ x = \{y: \varphi(y)\} \]

if and only if

\[
(\forall y)(y \in x \iff \varphi(y)).
\]

If there is such an \( x \), then \( x \) is unique up to \( \equiv \).

Let \( R, S \) be \( k \)-ary relations. The notion \( R \equiv S \) is usually too strong for our purposes.

DEFINITION 5.7.9. We define \( R \equiv' S \) if and only if

\[
(\forall x_1, \ldots, x_k)(R(x_1, \ldots, x_k) \iff S(x_1, \ldots, x_k)).
\]

DEFINITION 5.7.10. We define \( R \subseteq' S \) if and only if
We now prove comprehension for relations. To do this, we need a bounding lemma.

**LEMMA 5.7.4.** Let \( n, k \geq 1 \), and \( x_1, \ldots, x_k \leq d_n \). There exists \( y = \{ x_1, \ldots, x_k \} \) such that \( y \leq d_{n+1} \). There exists \( z = <x_1, \ldots, x_k> \) such that \( z \leq d_{n+1} \).

Proof: Let \( k, n, x_1, \ldots, x_k \) be as given. By Lemmas 5.7.1 and 5.7.3,

\[
(\exists y) (y = \{ x_1, \ldots, x_k \}) . \\
(\exists z) (z = <x_1, \ldots, x_k>).
\]

By Lemma 5.6.18 iii), let \( r > n \) be such that

\[
(\exists y \leq d_r) (y = \{ x_1, \ldots, x_k \}) . \\
(\exists z \leq d_r) (z = <x_1, \ldots, x_k>).
\]

By Lemma 5.6.18 v),

\[
(\exists y \leq d_{n+1}) (y = \{ x_1, \ldots, x_k \}) . \\
(\exists z \leq d_{n+1}) (z = <x_1, \ldots, x_k>).
\]

QED

**LEMMA 5.7.5.** Let \( k, n \geq 1 \) and \( \varphi(v_1, \ldots, v_{k+n}) \) be a formula of \( L^# \). Let \( y_1, \ldots, y_n, z \) be given. There is a \( k \)-ary relation \( R \) such that \( (\forall x_1, \ldots, x_k) (R(x_1, \ldots, x_k) \leftrightarrow (x_1, \ldots, x_k \leq z \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n))) \).

Proof: Let \( k, n, m, \varphi, y_1, \ldots, y_n, z \) be as given. By Lemma 5.6.18 iii), let \( r \geq 1 \) be such that \( y_1, \ldots, y_n, z \leq d_r \). By Lemma 5.6.18 iv), let \( R \) be such that

1) \( (\forall x) (x \in R \leftrightarrow (x \leq d_{r+1} \land (\exists x_1, \ldots, x_k \leq z \land (x = <x_1, \ldots, x_k> \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n))))). \)

Obviously \( R \) is a \( k \)-ary relation. We claim that

\[
(\forall x_1, \ldots, x_k) (R(x_1, \ldots, x_k) \leftrightarrow (x_1, \ldots, x_k \leq z \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n))).
\]

To see this, fix \( x_1, \ldots, x_k \). First assume \( R(x_1, \ldots, x_k) \). Let \( x = <x_1, \ldots, x_k>, x \in R \). By 1),

\[
(\forall x_1, \ldots, x_k) (R(x_1, \ldots, x_k) \leftrightarrow (x_1, \ldots, x_k \leq z \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n))).
\]
\[ x \leq d_{r+1} \land (\exists x_1^*, \ldots, x_k^* \leq z) (x = <x_1^*, \ldots, x_k^*> \land 
\varphi(x_1^*, \ldots, x_k^*, y_1, \ldots, y_n)). \]

Let \( x_1^*, \ldots, x_k^* \) be as given by this statement. By Lemma 5.7.3, \( x_1^* = x_1, \ldots, x_k^* = x_k \). Hence \( x_1, \ldots, x_k \leq z \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n). \)

Now assume
\[ x_1, \ldots, x_k \leq z \land \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n). \]

By Lemma 5.7.4, let
\[ x = <x_1, \ldots, x_k> \land x \leq d_{r+1}. \]

By 1), \( x \in R \). Hence \( R(x_1, \ldots, x_k) \). QED

**Lemma 5.7.6.** If \( x \approx \{y_1, \ldots, y_k\} \) then each \( y_1 < x \). If \( x = <y_1, \ldots, y_k>, k \geq 2 \), then each \( y_1 < x \). If \( x = <y_1, \ldots, y_k>, k \geq 1 \), then each \( y_1 \leq x \). If \( R(x_1, \ldots, x_k) \) then each \( x_i < R \).

**Proof:** The first claim is evident from Lemma 5.6.18 ii). The second claim is by external induction on \( k \geq 2 \). For the basis case \( k = 2 \), note that if \( x = <y, z> \) then \( y, z \) are both elements of elements of \( x \), and apply Lemma 5.6.18 ii). Now assume true for fixed \( k \geq 2 \). Let \( x = <y_1, \ldots, y_{k+1}>, \) and let \( z = <y_1, y_2>, x = <z, y_3, \ldots, y_{k+1}>, \) By induction hypothesis, \( z, y_3, \ldots, y_{k+1} < x, \) and also \( y_1, y_2 < x \).

The third claim involves only the new case \( k = 1 \), which is trivial.

For the final claim, let \( R(x_1, \ldots, x_k) \). Let \( x = <x_1, \ldots, x_k>, x \in R \). By the second claim and Lemma 5.6.18 iii), \( x_1, \ldots, x_k \leq x < R \). QED

**Definition 5.7.11.** A binary relation is defined to be a 2-ary relation. Let \( R \) be a binary relation. We "define"
\[
\begin{align*}
\text{dom}(R) & = \{x: (\exists y) (R(x, y))\}. \\
\text{rng}(R) & = \{x: (\exists y) (R(y, x))\}. \\
\text{fld}(R) & = \{x: (\exists y) (R(x, y) \lor R(y, x))\}.
\end{align*}
\]

Note that this constitutes a definition of \( \text{dom}(R) \), \( \text{rng}(R) \), \( \text{fld}(R) \) up to \( = \).
LEMMA 5.7.7. For all binary relations \( R \), \( \text{dom}(R) \) and \( \text{rng}(R) \) and \( \text{fld}(R) \) exist.

Proof: Let \( R \) be a binary relation. By Lemma 5.6.18 iv), let \( A, B, C \) be such that

\[
(\forall x)(x \in A \iff (x \leq R \land (\exists y)(R(x,y))).
\]

\[
(\forall x)(x \in B \iff (x \leq R \land (\exists y)(R(y,x))).
\]

\[
(\forall x)(x \in C \iff (x \leq R \land (\exists y)(R(x,y) \lor R(y,x))).
\]

By Lemma 5.7.6,

\[
(\forall x)(x \in A \iff (\exists y)(R(x,y)).
\]

\[
(\forall x)(x \in B \iff (\exists y)(R(y,x)).
\]

\[
(\forall x)(x \in C \iff (\exists y)(R(x,y) \lor R(y,x))).
\]

QED

DEFINITION 5.7.12. A pre well ordering is a binary relation \( R \) such that

i) \((\forall x \in \text{fld}(R))(R(x,x));\)

ii) \((\forall x,y,z \in \text{fld}(R))(R(x,y) \land R(y,z)) \rightarrow R(x,z));\)

iii) \((\forall x,y \in \text{fld}(R))(R(x,y) \lor R(y,x));\)

iv) \((\forall x \subseteq \text{fld}(R))(\neg(x = \emptyset) \rightarrow (\exists y \in x)(\forall z \in x)(R(y,z))).\)

Note that \( R \) is a pre well ordering if and only if \( R \) is reflexive, transitive, connected, and every nonempty subset of its field (or domain) has an \( R \) least element.

Note that all pre well orderings are reflexive. Clearly for pre well orderings \( R \), \( \text{dom}(R) = \text{rng}(R) = \text{fld}(R).\)

Let \( R \) be a reflexive and transitive relation.

DEFINITION 5.7.13. It will be convenient to write \( R(x,y) \) as \( x \leq_R y \), and write \( x =_R y \) for \( x \leq_R y \land y \leq_R x \). We also define \( x \geq_R y \leftrightarrow y \leq_R x, x <_R y \leftrightarrow x \leq_R y \land \neg y \leq_R x, x >_R y \leftrightarrow y <_R x, \) and \( x \neq_R y \leftrightarrow \neg x =_R y.\)

DEFINITION 5.7.14. Let \( R \) be a pre well ordering and \( x \in \text{fld}(R) \). We "define" the binary relations \( R|<x \) by

\[
(\forall y,z)(R|<x(y,z) \iff y \leq_R z <_R x)).
\]
Note that $R|<x$ is unique up to $\equiv'$. Also note that by Lemma 5.7.5, $R|<x$ exists. Furthermore, it is easy to see that $R|<x$ is a pre well ordering.

When we write $R|<x$, we require that $x \in \text{fld}(R)$.

**DEFINITION 5.7.15.** Let $R, S$ be pre well orderings. We say that $T$ is an isomorphism from $R$ onto $S$ if and only if

i) $T$ is a binary relation;
ii) $\text{dom}(T) \equiv \text{dom}(R)$, $\text{rng}(T) \equiv \text{dom}(S)$;
iii) Let $T(x, y)$, $T(z, w)$. Then $x \leq_R z \iff y \leq_S w$;
iv) Let $x =_R u$, $y =_S v$. Then $T(x, y) \iff T(u, v)$.

**LEMMA 5.7.8.** Let $R, S$ be pre well orderings, and $T$ be an isomorphism from $R$ onto $S$. Let $T(x, y)$, $T(z, w)$. Then $x <_R z \iff y <_S w$, and $x =_R z \iff y =_S w$.

**Proof:** Let $R, S, T, x, y, z, w$ be as given. Suppose $x <_R z$. Then $y \leq_S w$. If $w \leq_S y$ then $z \leq_R x$. Hence $y <_R w$. Suppose $y <_S w$. Then $x \leq_R z$. If $z \leq_R x$ then $w \leq_S y$. Hence $x <_R z$. Suppose $x =_R z$. Then $y \leq_S w$ and $w \leq_S y$. Hence $y =_S w$. Suppose $y =_S w$. Then $x \leq_R z$ and $z \leq_R x$. Hence $x =_R z$. QED

**LEMMA 5.7.9.** Let $R, S$ be pre well orderings. Let $a, b \in \text{dom}(S)$. Let $T$ be an isomorphism from $R$ onto $S|<a$, and $T^*$ be an isomorphism from $R$ onto $S|<b$. Then $a =_S b$ and $T \equiv' T^*$.

**Proof:** Let $R, S, a, b, T, T^*$ be as given. Suppose there exists $x \in \text{dom}(R)$ such that for some $y$, $\neg(T(x, y) \iff T^*(x, y))$. By Lemma 5.6.18 iv), let $x$ be $R$ least with this property.

case 1. $(\exists y)(T(x, y) \land \neg T^*(x, y))$. Let $T(x, y)$, $\neg T^*(x, y)$. Also let $T^*(x, y^*)$. If $y =_S y^*$ then by clause iv) in the definition of isomorphism, $T^*(x, y)$. Hence $\neg y =_S y^*$.

case 1a. $y <_S y^*$. Then $y <_S b$. Let $T^*(x^*, y^*)$.

Suppose $x^* <_R x$. If $\neg T(x^*, y)$, then we have contradicted the choice of $x$. Hence $T(x^*, y)$. But this contradicts $T(x, y)$ by Lemma 5.7.8.

Suppose $x \leq_S x^*$. By $T^*(x, y^*)$, $T^*(x^*, y)$ and Lemma 5.7.8, $y^* \leq_S y$. This is a contradiction.

case 1b. $y^* <_S y$. Then $y^* <_S a$. Let $T(x^*, y^*)$. By $T(x, y)$ and Lemma 5.7.8, $x^* <_R x$. By the choice of $x$, since $T(x^*, y^*)$, we
have $T^*(x^*, y^*)$. By Lemma 5.7.8, since $T^*(x, y^*)$, we have $x =_R x^*$. Since $T(x, y)$, by Lemma 5.7.8 we have $y =_S y^*$. This is a contradiction.

case 2. $(\exists y) (\neg T(x, y) \land T^*(x, y))$. Let $\neg T(x, y), T^*(x, y)$. This is the same as case 1, interchanging $a, b$, and $T, T^*$.

We have now established that $T \nsimeq T^*$. If $a <_S b$ then $a \in \text{rng}(T^*)$ but $b \notin \text{rng}(T)$. This contradicts $T \nsimeq T^*$. If $b <_S a$ then $b \in \text{rng}(T)$ but $b \notin \text{rng}(T^*)$. This also contradicts $T \nsimeq T^*$. Therefore $a =_S b$. QED

**DEFINITION 5.7.16.** Let $R, S$ be pre well orderings. Let $T$ be an isomorphism from $R$ onto $S$. Let $x \in \text{dom}(R)$. We write $T|<x$ for "the" restriction of $T$ to first arguments $u <_R x$. We write $T|<x$ for "the" restriction of $T$ to first arguments $u <_R x$. Note that $T|<x$, $T|<x$ are each unique up to $\equiv'$.

**LEMMA 5.7.10.** Let $R, S$ be pre well orderings. Let $T$ be an isomorphism from $R$ onto $S$, and $T(x, y)$. Then $T|<x$ is an isomorphism from $R|<x$ onto $S|<y$.

Proof: Let $R, S, T, x, y$ be as given. It suffices to show that $\text{rng}(T|<x) = \{ b : b <_S y \}$. Let $b <_S y$. Let $T(a, b)$. By Lemma 5.7.8, $a <_R x$. Hence $b \in \text{rng}(T|<x)$. QED

**LEMMA 5.7.11.** Let $R, S$ be pre well orderings, $T$ be an isomorphism from $R$ onto $S$, and $T^*$ be an isomorphism from $R|<x$ onto $S|<y$. Then $T^* \equiv T|<x$ and $T(x, y)$.

Proof: Let $R, S, T, T^*, x, y$ be as given. Let $T(x, y^*)$. By Lemma 5.7.10, $T|<x$ is an isomorphism from $R|<x$ onto $S|<y^*$. By Lemma 5.7.9, $y =_S y^*$ and $T|<x \equiv T^*$. Hence $T(x, y)$. QED

**DEFINITION 5.7.17.** Let $T$ be a binary relation. We write $T^{-1}$ for the binary relation given by $T^{-1}(x, y) \iff T(y, x)$. By Lemma 5.7.5, $T^{-1}$ exists. Obviously $T^{-1}$ is unique up to $\equiv'$.

**LEMMA 5.7.12.** Let $R, S$ be pre well orderings, and $T$ be an isomorphism from $R$ onto $S$. Then $T^{-1}$ is an isomorphism from $S$ onto $R$.

Proof: Let $R, S, T$ be as given. Obviously $\text{dom}(T^{-1}) = \text{dom}(S)$ and $\text{rng}(T^{-1}) = \text{dom}(R)$. Let $T^{-1}(x, y), T^{-1}(z, w)$. Then $T(y, x), T(w, z)$. Hence $y =_R z \iff x =_S z$. 
Finally, let $T^{-1}(x,y)$, $x =_R u$, $y =_S v$. Then $T(y,x)$, $T(v,u)$, $T^{-1}(u,v)$. QED

**Definition 5.7.18.** Let $R$ be a pre well ordering. We can append a new point \( \infty \) on top and form the extended pre well ordering $R^\prime$. The canonical way to do this is to use $R$ itself as the new point. This defines $R^\prime$ uniquely up to $\equiv$.

Clearly $R^\prime|< \infty \equiv' R$.

**Lemma 5.7.13.** Let $R, S$ be pre well orderings. Exactly one of the following holds.
1. $R, S$ are isomorphic.
2. $R$ is isomorphic to some $S|< y$, $y \in \text{dom}(S)$.
3. Some $R|< x$, $x \in \text{dom}(R)$, is isomorphic to $S$.

In case 2, the $y$ is unique up to $=_S$. In case 3, the $x$ is unique up to $=_R$. In all three cases, the isomorphism is unique up to $\equiv'$.

Proof: We first prove the uniqueness claims. For case 1, let $T, T^*$ be isomorphisms from $R$ onto $S$. Then $T, T^*$ are isomorphisms from $R$ onto $S^\prime|< \infty$. By Lemma 5.7.9, $T \equiv' T^*$.

For case 2, Let $T$ be an isomorphism from $R$ onto $S|< y$, and $T^*$ be an isomorphism from $R$ onto $S|< y^*$. Apply Lemma 5.7.9.

For case 3, Let $T$ be an isomorphism from $R|< x$ onto $S$, and $T^*$ be an isomorphism from $R|< x^*$ onto $S$. By Lemma 5.7.12, $T^{-1}$ is an isomorphism from $S$ onto $R|< x$, and $T^*-1$ is an isomorphism from $S$ onto $R|< x^*$. Apply Lemma 5.7.9.

For uniqueness, it remains to show that at most one case applies. Suppose cases 1,2 apply. Let $T$ be an isomorphism from $R$ onto $S$, and $T^*$ be an isomorphism from $R$ onto $S|< y$. Then $T$ is an isomorphism from $R$ onto $S^\prime|< \infty$, and $T^*$ is an isomorphism from $R$ onto $S^\prime|< y$. By Lemma 5.7.9, $y$ is $\infty$, which is a contradiction.

Suppose cases 1,3 hold. Let $T$ be an isomorphism from $R$ onto $S$, and $T^*$ be an isomorphism from $R|< x$ onto $S$. Then $T^{-1}$ is an isomorphism from $S$ onto $R^\prime|< \infty$, and $T^*-1$ is an isomorphism from $S$ onto $R^\prime|< x$. By Lemma 5.7.9, $x$ is $\infty$, which is a contradiction.

Suppose cases 2,3 hold. Let $T$ be an isomorphism from $R$ onto $S|< y$ and $T^*$ be an isomorphism from $R|< x$ onto $S$. By Lemma 5.7.10, $T|< x$ is an isomorphism from $R|< x$ onto $S|< z$, where
T(x,z). Hence T|x is an isomorphism from R|x onto S'|<z.
Also T* is an isomorphism from R|x onto S'|<∞. Hence by Lemma 5.7.9, z is ∞. This is a contradiction.

We now show that at least one of 1-3 holds. Consider all isomorphisms from some R'|<x onto some S'|<y, x ∈ dom(R'), y ∈ dom(S'). We call these the local isomorphisms.

We claim the following, concerning these local isomorphisms. Let T be an isomorphism from R'|<x onto S'|<y, and T* be an isomorphism from R'|<x* onto S'|<y*. If x = R+ x* then y = S+ y* and T ≡ T*. If x < R+ x* then y < S+ y* and T ≡ T*|<x. If x* < R+ x then y* < S+ y and T* ≡ T|<x*.

To see this, let T,T*,x,y be as given.

case 1. x = R+ x*. Apply Lemma 5.7.9.

case 2. x* < R+ x. Suppose y ≤ S+ y*. Let T(x*,z), z ≤ S+ y. By Lemma 5.7.10, T|x* is an isomorphism from R'|<x* onto S'|<z. By Lemma 5.7.9, T* ≡ T|x* and z = S+ y*. This is a contradiction. Hence y* < S+ y. By Lemma 5.7.10, T|x* is an isomorphism from R'|<x* onto S'|<w, where T(x*,w), w < S+ y. By Lemma 5.7.9, T* ≡ T|x*.

case 3. x < R+ x*. Symmetric to case 2.

By Lemma 5.7.5, we can form the union T of all of the local isomorphisms, since the underlying arguments are all in dom(R') or dom(S'), both of which are bounded.

By the pairwise compatibility of the local isomorphisms, T obeys conditions iii), iv) in the definition of isomorphism. It is also clear that the domain of T is closed downward in R', and the range of T is closed downward in S'. Hence dom(T) = {u: u < R+ x}, rng(T) = {v: v < S+ y}, for some x ∈ dom(R'), y ∈ dom(S'). Hence T is an isomorphism from R'|<x onto S'|<y.

We now argue by cases.

case 1. x,y are ∞. Then T is an isomorphism from R onto S.

case 2. x is ∞, y ∈ dom(S). Then T is an isomorphism from R onto S|<y*, y* defined below.
case 3. $x \in \text{dom}(R)$, $y$ is $\infty$. Then $T$ is an isomorphism from $R|<x^*$ onto $S$, $x^*$ defined below.

case 4. $x \in \text{dom}(R), y \in \text{dom}(S)$. Then $T$ is an isomorphism from $R|<x$ onto $S|<y$. Using Lemma 5.7.5, let $T*$ be defined by

$$T^*(u,v) \leftrightarrow T(u,v) \vee (u =_R x \land v =_S y).$$

Then $T^*$ is an isomorphism from $R|<x^*$ onto $S|<y^*$, where $x^*,y^*$ are respective immediate successors of $x,y$ in $R^+,S^+$. This contradicts the definition of $T$. QED

**Lemma 5.7.14.** Let $R,S,S^*$ be pre well orderings. Let $T$ be an isomorphism from $R$ onto $S$, and $T^*$ be an isomorphism from $S$ onto $S^*$. Define $T^{**}(x,y) \leftrightarrow (\exists z)(T(x,z) \land T^*(z,y))$, by Lemma 5.7.5. Then $T^{**}$ is an isomorphism from $R$ onto $S^*$.

**Proof:** Let $R,S,S^*,T,T^*,T^{**}$ be as given. Note that $T^{**}$ is defined up to $\equiv'$. Obviously $\text{dom}(T^{**}) \equiv \text{dom}(R)$, $\text{rng}(T^{**}) \equiv \text{dom}(S^*)$.

Suppose $T^{**}(x,y)$, $T^{**}(x^*,y^*)$. Let $T(x,z)$, $T^*(z,y)$, $T(x^*,w)$, $T^*(w,y^*)$. Then $x \leq_R x^* \iff z \leq_S w$, $z \leq_R w \iff y \leq_S y^*$. Therefore $x \leq_R x^* \iff y \leq_S y^*$.

Suppose $T^{**}(x,y)$, $x =_R u$, $y =_S v$. Let $T(x,z)$, $T^*(z,y)$. Then $T(u,z)$, $T^*(z,v)$. Hence $T^{**}(u,v)$. QED

We introduce the following notation in light of Lemma 5.7.13.

**Definition 5.7.19.** Let $R,S$ be pre well orderings. We define

$$R \equiv S \leftrightarrow R,S \text{ are pre well orderings and } R,S \text{ are isomorphic.}$$

$$R < S \leftrightarrow R \leq S \land R \not\equiv S.$$
equivalence relation on pre well orderings. \( \leq^{**} \) is reflexive and transitive and connected on pre well orderings. Let \( R, S, S^* \) be pre well orderings. (\( R \leq^{**} S \land S <^{**} S^* \)) \( \rightarrow \) \( R <^{**} S^* \). (\( R <^{**} S \land S \leq^{**} S^* \)) \( \rightarrow \) \( R <^{**} S^* \). \( R <^{**} S \lor S \leq^{**} S^* \rightarrow R \leq^{**} S \), with exclusive \( \lor \). \( R \leq^{**} S \lor S \leq^{**} R \). (\( R \leq^{**} S \land S \leq^{**} R \)) \( \rightarrow \) \( R =^{**} S \).

Proof: We apply Lemmas 5.7.13 and 5.7.14. For the first claim, if \( R <^{**} S \) then we are in case 2 of Lemma 5.7.13, and the \( y \) is unique up to \( =_s \).

For the second claim, \( <^{**} \) is irreflexive since \( R <^{**} R \) implies that cases 1, 2 both hold in Lemma 5.7.13 for \( R, R \). Also, suppose \( R <^{**} S, S <^{**} S^* \). Let \( T \) be an isomorphism from \( R \) onto \( S|<y \), and \( T^* \) be an isomorphism from \( S \) onto \( S^*|<z \). By Lemma 5.7.10, Let \( T^{**} \) be an isomorphism from \( S|<y \) onto \( S^*|<w \). By Lemma 5.7.14, there is an isomorphism from \( R \) onto \( S^*|<w \). Hence \( R <^{**} S^* \).

For the third claim, note that \( R =^{**} R \) because there is an isomorphism from \( R \) onto \( R \) by defining \( T(x, y) \leftrightarrow x =_r y \). Now suppose \( R =^{**} S \), and let \( T \) be an isomorphism from \( R \) onto \( S \). By Lemma 5.7.12, \( T^{-1} \) is an isomorphism from \( S \) onto \( R \). Hence \( S =^{**} R \). Finally, suppose \( R =^{**} S, S =^{**} S^* \), and let \( T \) be an isomorphism from \( R \) onto \( S \), \( T^* \) be an isomorphism from \( S \) onto \( S^* \). By Lemma 5.7.14, \( R =^{**} S^* \).

For the fourth claim, since \( R =^{**} R \), we have \( R \leq^{**} R \). For transitivity, let \( R \leq^{**} S, S \leq^{**} S^* \). If \( R <^{**} S, S <^{**} S^* \), then by the second claim, \( R <^{**} S^* \), and so \( R \leq^{**} S^* \). If \( R =^{**} S, S =^{**} S^* \), then by Lemma 5.7.14, \( R =^{**} S^* \), and so \( R \leq^{**} S^* \). The remaining two cases for transitivity follow from the fifth and sixth claims. Connectivity of \( \leq^{**} \) is by Lemma 5.7.13.

For the fifth claim, let \( R \leq^{**} S \) and \( S <^{**} S^* \). By the second claim, we have only to consider the case \( R =^{**} S \). Let \( S \) be isomorphic to \( S^*|<y \). Since \( R \) is isomorphic to \( S \), by the third claim, \( R \) is isomorphic to \( S^*|<y \). Hence \( R <^{**} S^* \).

For the sixth claim, let \( R <^{**} S \) and \( S \leq^{**} S^* \). By the second claim, we have only to consider the case \( S =^{**} S^* \). Let \( R \) be isomorphic to \( S|<y \). By Lemma 5.7.10, \( S|<y \) is isomorphic to \( S^*|<z \), for some \( z \in \text{dom}(S^*) \). By the third claim, \( R \) is isomorphic to \( S^*|<z \). Hence \( R <^{**} S^* \).
The seventh and eighth claims are immediate from Lemmas 5.7.12 and 5.7.13.

For the ninth claim, let \( R \leq S \) and \( S \leq R \). Assume \( R <** S \). By the sixth claim \( R <** R \), which is a contradiction. Assume \( S <** R \). By the sixth claim, \( S <** S \), which is also a contradiction. By the eighth claim, \( R \leq S \lor S \leq R \). Under either disjunct, \( R =** S \). QED

**Lemma 5.7.16.** Every nonempty set of pre well orderings has a \( \leq ** \) least element.

**Proof:** Let \( A \) be a nonempty set of pre well orderings, and fix \( S \in A \). We can assume that there exists \( R \in A \) such that \( R <** S \), for otherwise, \( S \) is a \( \leq ** \) minimal element of \( A \).

By Lemma 5.7.5, define

\[ B = \{ y \in \text{dom}(S): (\exists R \in A) (T =** S|<y) \}. \]

Let \( y \) be an \( S \) least element of \( B \). Let \( R \in A \) be isomorphic to \( S|<y \).

We claim that \( R \) is a \( \leq ** \) least element of \( A \). To see this, by trichotomy, let \( R^* <** R, R^* \in A \). Then \( R^* <** S|<y \), since \( R \) is isomorphic to \( S|<y \).

Let \( R^* \) be isomorphic to \( (S|<y)|<z, z < S \ y \). Then \( R^* \) is isomorphic to \( S|<z, z < S \ y \). This contradicts the choice of \( y \). QED

**Definition 5.7.20.** For \( x,y \in D \), we define \( x <# y \) if and only if

there exists a pre well ordering \( S \leq y \) such that

for every pre well ordering \( R \leq x \), \( R <** S \).

We caution the reader that the \( \leq \) in the above definition is not to be confused with \( \leq ** \). It is from the \( < \) of \( D \) in the structure \( M# \). In particular, \( x,y \) generally will not be pre well orderings. Thus here we are treating \( R,S \) as points.

**Definition 5.7.21.** We define \( x \leq # y \) if and only if

for all pre well orderings \( R \leq x \) there exists a pre well ordering \( S \leq y \) such that \( R \leq ** S \).
LEMMA 5.7.17. <# is an irreflexive and transitive relation on D. Let x, y ∈ D. x ≤# y → x ≤ y. (x ≤# y ∧ y <# z) → x <# z. (x <# y ∧ y ≤# z) → x ≤ y. x ≤ y → x <# y. x <# y ⇔ ¬y <# x. x ≤ y ⇔ ¬y ≤ x.

Proof: For the first claim, <# is irreflexive since <** is irreflexive. Suppose x <# y and y <# z. Let S ≤ y be a pre well ordering such that for all pre well orderings R ≤ x, R <** S. Let S* ≤ z be a pre well ordering such that for all pre well orderings R ≤ y, R <** S*. Then S <** S*. Hence for all pre well orderings R ≤ x, R <** S <** S*. Hence for all pre well orderings R ≤ x, R <** S*, by the transitivity of <**. Since S* ≤ z, we have x ≤# z.

For the second claim, x ≤# x since ≤** on pre well orderings is reflexive. Suppose x ≤# y and y ≤# z. Let R ≤ x. Let S ≤ y, R ≤** S. Let S* ≤ z, S ≤** S*. By the transitivity of ≤**, R ≤** S*.

For the third claim, let ¬(x ≤# y). Let R ≤ x be a pre well ordering such that for all pre well orderings S ≤ y, we have ¬R ≤** S. We claim that y <# x. To see this, let S ≤ y be a pre well ordering. Then ¬R ≤** S. By Lemma 5.7.15, S <** R.

For the fourth claim, let x <# y. Let S ≤ y be a pre well ordering such that for all pre well orderings R ≤ x, R <** S. Let R ≤ x be a pre well ordering. Then R ≤** S. Hence x ≤# y.

For the fifth claim, let x ≤# y and y <# z. Let S ≤ z be a pre well ordering such that for all pre well orderings R ≤ y, R <** S. Let R ≤ x be a pre well ordering. Let S* ≤ y be a pre well ordering such that R ≤** S*. Then S* <** S. By Lemma 5.7.15, R <** S. We have verified that x <# z.

For the sixth claim, let x <# y and y ≤# z. Let S ≤ y be a pre well ordering such that for all pre well orderings R ≤ x, R <** S. Let S* ≤ z be a pre well ordering such that S ≤** S*. By Lemma 5.7.15, for all pre well orderings R ≤ x, R <** S*. Hence x <# z.

The seventh claim is obvious.

For the eight claim, let x <# y. Let S ≤ y be a pre well ordering, where for all pre well orderings R ≤ x, we have R
<** S. If \( y \leq x \) then \( S \leq x \), and so \( S <** S \). This is a contradiction. Hence \( x < y \).

For the ninth claim, the converse is the first claim. Suppose \( x \# y \land y <# x \). By the third claim, \( x <# x \), which is impossible.

For the tenth claim, the converse is the first claim. Suppose \( x <# y \land y \leq # x \). By the third claim, \( y <# y \), which is impossible. QED

We now define \( x =# y \) if and only if \( x \leq # y \land y \leq # x \).

**LEMMA 5.7.18.** =# is an equivalence relation on D. Let \( x, y \in D \). \( x \leq # y \iff (x <# y \lor x =# y) \). \( x <# y \lor y <# x \lor x =# y \), with exclusive \( \lor \).

**Proof:** For the first claim, reflexivity and symmetry are obvious, by Lemma 5.7.17. Let \( x =# y \) and \( y =# z \). Then \( x \leq # y \) and \( y \leq # z \). Hence \( x \leq # z \). Also \( z \leq # y \) and \( y \leq # x \). Hence \( z \leq # x \). Therefore \( x =# z \).

For the second claim, let \( x, y \in D \). By Lemma 5.7.17, \( x \leq # y \lor y <# x \). By the first claim, \( x <# y \lor y <# x \lor x =# y \).

To see that the \( \lor \) is exclusive, suppose \( x <# y \), \( y <# x \). By Lemma 5.7.17, \( x <# x \), which is a contradiction. Suppose \( x <# y \), \( x =# y \). By Lemma 5.7.17, \( x <# x \), which is a contradiction. Suppose \( y <# x \), \( x =# y \). By Lemma 5.7.17, \( y <# y \), which is a contradiction. QED

**DEFINITION 5.7.22.** We say that \( S \) is \( x \)-critical if and only if

i) \( S \) is a pre well ordering;
ii) for all pre well orderings \( R \leq x \), \( R <** S \);
iii) for all \( y \in \text{dom}(S) \), \( S|<y \leq** \) some pre well ordering \( R \leq x \).

**LEMMA 5.7.19.** Assume \((\forall y \in x)(y \text{ is a pre well ordering})\). Then there exists a pre well ordering \( S \) such that \((\forall R \in x)(R <** S) \land (\forall u \in \text{dom}(S))(\exists R \in x)(S|<u <** R)\).

**Proof:** Let \( x \) be as given. Let \( x < d_r, r \geq 1 \). By Lemma 5.7.20 iv), define

\[ E = \{ y \leq d_{r+1} : \]
\[(\exists R, z)(R \in x \land y \text{ is an } R|<z)\].

By Lemma 5.7.5, we define
\[S(u,v) \leftrightarrow u, v \in E \land u \leq^* v.\]

Then \(S\) is uniquely defined up to \(\equiv\'). By Lemmas 5.7.15, 5.7.16, \(S\) is a pre well ordering.

Let \(R \in x\) and \(z \in \text{dom}(R)\). By Lemma 5.6.18 iv),
\[(\exists y)(y \text{ is an } R|<z).\]

By Lemma 5.6.18 iii), let \(p \geq r+1\) be such that
\[(\exists y < d_p)(y \text{ is an } R|<z).\]

By Lemma 5.7.20 v),
\[(\exists y < d_{r+1})(y \text{ is an } R|<z).\]

Hence every \(R|<z, R \in x,\) is isomorphic to an element of \(E\).

We claim that we can define an isomorphism \(T_R\) from any given \(R \in x,\) onto \(S\) or a proper initial segment of \(S,\) as follows. \(T_R\) relates each \(z \in \text{dom}(R)\) to the elements of \(E\) that are isomorphic to \(R|<z.\) Note that each \(z \in \text{dom}(R)\) gets related by \(T_R\) to something; i.e., all of the \(R|<z\) lying in \(E.\)

To verify the claim, we first show that \(\text{rng}(T_R)\) is closed downward under \(\leq^*\) in \(E.\) Fix \(T_R(z,w).\) Let \(w^*\) be an \(S\) least element of \(E,\) \(w^* <^* w,\) which is not in \(\text{rng}(T_R)\). Then \(T_R\)
must act as an isomorphism from some proper initial segment \(J\) of \(R|<z\) onto \(S|<w^*\). We can assume \(J \in E\) (by taking an isomorphic copy). Hence \(T_R(J,w^*),\) contradicting that \(w^* \notin \text{rng}(T_R)\).

Since \(\text{rng}(T_R)\) is closed downward under \(\leq^*\) in \(E,\) we see that \(\text{rng}(T_R) = E,\) or \(\text{rng}(T_R) = S|<v,\) for some \(v \in E.\) From the definition of \(T_R,\) \(T_R\) is an isomorphism from \(R\) onto \(S\) or a proper initial segment of \(S.\) Hence \(R \leq^* S.\)

Now let \(u \in \text{dom}(S).\) Then \(u\) is some \(R|<z, R \in x.\) Therefore \(u <^* R,\) for some \(R \in x.\) QED
LEMMA 5.7.20. Assume $(\forall y \in x)(y \text{ is a pre well ordering}). Then there exists a pre well ordering $S$ such that $(\forall R \in x)(R \prec S) \land (\forall R \prec S)(\exists y \in x)(R \preceq y)$.

Proof: Let $x$ be as given.

case 1. $x$ has a $\preceq$ greatest element $R$. Set $S \equiv R^\dagger$.

case 2. Otherwise. Set $S$ to be as provided by Lemma 5.7.19 applied to $x$.

QED

LEMMA 5.7.21. For all $x$, there exists an $x$-critical $S$. If $S$ is $x$-critical then $x \prec S$.

Proof: Let $x$ be given. By Lemma 5.6.18 iv), define

$$x^* = \{R: R \leq x \land R \text{ is a pre well ordering}\}.$$ 

Let $S$ be as provided by Lemma 5.7.20. Then $S$ is $x$-critical.

Now let $S$ be $x$-critical. If $S \leq x$ then $S \prec S$, which is impossible by ii) in the definition of $x$-critical. QED

LEMMA 5.7.22. For all $x$, all $x$-critical $S$ are isomorphic. For all $x,y$, $x \prec# y$ if and only if $(\exists R,S)(R \text{ is } x\text{-critical} \land S \text{ is } y\text{-critical} \land R \prec S)$.

Proof: Let $R,S$ be $x$-critical. Suppose $R \prec S$, and let $R = S\prec y$. By clause iii) in the definition of $x$-critical, let $S\prec y \preceq S$, $R \text{ a pre well ordering}. By clause ii) in the definition of $R$ is $x$-critical, $R \prec S$. Hence $R \preceq S$. This is a contradiction. Hence $(R \prec S)$. By symmetry, we also obtain $(S \prec R)$. Hence $R,S$ are isomorphic.

For the second claim, let $x,y \in D$. First assume $x \prec# y$. Let $R$ be $x$-critical and $S$ be $y$-critical. Let $S^\dagger \leq y$ be a pre well ordering such that for all pre well orderings $R^\dagger \leq x$, we have $R^\dagger \prec S^\dagger$.

We claim that $R \preceq S^\dagger$. To see this, suppose $S^\dagger \prec R$, and let $S^\dagger$ be isomorphic to $R\prec z$. Since $R$ is $x$-critical, let $R\prec z \preceq R \preceq x$, where $R$ is a pre well ordering. Then $S^\dagger \preceq R^\dagger$. Since $R^\dagger \preceq x$, we have $R^\dagger \prec S^\dagger$, which is a contradiction. Thus $R \preceq S^\dagger$. 
Note that $S^* <** S$ since $S^* \leq y$ and $S$ is $y$-critical. Hence $R <** S$.

For the converse, assume $R$ is $x$-critical, $S$ is $y$-critical, and $R <** S$. Let $R$ be isomorphic to $S|<z$. Since $S$ is $y$-critical, let $S|<z S^* \leq y$, where $R^*$ is a pre well ordering. Then $R \leq** R^* \leq y$.

We claim that for all pre well orderings $S^* \leq x$, $S^* <** R^*$. To see this, let $S^* \leq x$ be a pre well ordering. Since $R$ is $x$-critical, $S^* <** R \leq** R^* \leq y$.

We have shown that $x <\# y$ using $R^* \leq y$, as required. QED

**Lemma 5.7.23.** Let $n \geq 1$. For all $x \leq d_n$ there exists $x$-critical $S < d_{n+1}$. $d_n <\# d_{n+1}$.

**Proof:** Let $n \geq 1$ and $x \leq d_n$. By Lemmas 5.7.21 and 5.6.18 ii), there exists $m > n$ such that the following holds.

$$\exists S < d_m)(S \text{ is } x\text{-critical}).$$

By Lemma 5.6.18 v),

$$\exists S < d_{n+1})(S \text{ is } x\text{-critical}).$$

For the second claim, by the first claim let $R < d_{n+1}$, where $R$ is $d_n$-critical. Let $S$ be $d_{n+1}$-critical. Then $R <** S$. By Lemma 5.7.22, $d_n <\# d_{n+1}$. QED

**Lemma 5.7.24.** If $y \in x$ then $x$ has a $<\#$ least element. Every first order property with parameters that holds of some $x$, holds of a $<\#$ least $x$. 0 is a $<\#$ least element.

**Proof:** Let $y \in x$. By Lemma 5.6.18 ii), let $n \geq 1$ be such that $x \leq d_n$. By Lemma 5.7.23, for each $y \in x$ there exists a $y$-critical $S < d_{n+1}$. By Lemma 5.6.18 iv), we can define

$$B = \{S < d_{n+1} : (\exists y \in x)(S \text{ is } y\text{-critical})\}$$

uniquely up to $=$.

By Lemma 5.7.16, let $S$ be a $<**$ least element of $B$. Let $S$ be $y$-critical, $y \in x$. We claim that $y$ is a $<\#$ minimal element of $x$. Suppose $z <\# y$, $z \in x$. By Lemma 5.7.23, let $R$ be $z$-critical, $R \in B$. By the choice of $S$, $S \leq** R$. By Lemma
5.7.22, let $R^*, S^*$ be such that $R^*$ is z-critical, $S^*$ is y-critical, and $R^* <\leftrightarrow^* S^*$. By the first claim of Lemma 5.7.22, $R <\leftrightarrow^* S$. This is a contradiction.

For the second claim, let $\varphi(y)$. By Lemma 5.6.18 ii), let $y < d_n$. By Lemma 5.6.18 iv), let $x = \{y < d_{n+1}: \varphi(y)\}$. By the first claim, let $y$ be a $<_\#$ minimal element of $x$. Suppose $\varphi(z), z <\# y$. Since $z \notin x$, we have $z \geq d_{n+1}$. Since $z <\# y$, we have $z < y$ (Lemma 5.7.17). This contradicts $y < d_{n+1} \wedge z \geq d_{n+1}$.

The third claim follows immediately from the last claim of Lemma 5.7.17. QED

**Lemma 5.7.25.** If $x \leq y$ then $x \leq\# y$. If $x \leq y \leq z$ and $x =\# z$, then $x =\# y =\# z$.

**Proof:** The first claim is trivial.

For the second claim, let $x \leq y \leq z$, $x =\# z$. Using the first claim and Lemmas 5.7.17, 5.7.18, $x \leq\# y \leq\# z \leq\# x$. Hence $x =\# y =\# z$. QED

From Lemma 5.7.25, we obtain a picture of what $<_\#$ looks like.

**Lemma 5.7.26.** $=\#$ is an equivalence relation on $D$ whose equivalence classes are nonempty intervals in $D$ (not necessarily with endpoints). These are called the intervals of $=\#$, $x <\# y$ if and only if the interval of $=\#$ in which $x$ lies is entirely below the interval of $=\#$ in which $y$ lies. There is no highest interval for $=\#$. The d’s lie in different intervals of $=\#$, each entirely higher than the previous.

**Proof:** For the first claim, $=\#$ is an equivalence relation by Lemma 5.7.18. Suppose $x < y$, $x =\# y$. By Lemma 5.7.25, any $x < z < y$ has $x =\# z =\# y$. So the equivalence classes under $=\#$ are intervals in $<$. 

For the second claim, let $x <\# y$. Let $z$ lie in the same interval of $=\#$ as $x$. Let $w$ lie in the same interval of $=\#$ as $y$. Then $x =* z, y =* w$. By Lemma 5.7.18, $z <\# w$. By Lemma 5.7.17, $z < w$.

Conversely, assume the interval of $=\#$ in which $x$ lies is entirely below the interval of $=\#$ in which $y$ lies. Then $\neg(x <\# y)$.
By Lemma 5.7.18, \( x <# y \lor y <# x \). The later implies \( y < x \), which is impossible. Hence \( x <# y \).

For the final claim, by Lemma 5.7.23, each \( d_n <# d_{n+1} \). By the second claim, the intervals of \( =# \) in which \( d_n \) lies is entirely below the interval of \( =# \) in which \( d_{n+1} \) lies. QED

Recall the component NAT in the structure \( M# \).

**Lemma 5.7.27.** There is a binary relation RNAT (recursively defined natural numbers) such that

i) \( \text{dom}(\text{RNAT}) = \{x: \text{NAT}(x)\} \);

ii) \( (\forall y)(\text{RNAT}(0, y) \leftrightarrow y \text{ is a } <# \text{ least element}) \);

iii) \( (\forall x)(\text{NAT}(x) \rightarrow (\forall w)(\text{RNAT}(x+1, w) \leftrightarrow (\exists z)(\text{RNAT}(x, z) \land w \text{ is an immediate successor of } z \text{ in } <#))) \);

iv) \( \text{RNAT} < d_2 \).

Any two RNAT’s (even without iv) are \( \equiv' \). If \( \text{NAT}(x) \) then \( \{y: \text{RNAT}(x, y)\} \) forms an equivalence class under \( =# \).

**Proof:** We will use the following facts. The set of all \( <# \) minimal elements exists and is nonempty. For all \( y \), the set of all immediate successors of \( y \) in \( <# \) exists and is nonempty. These follow from Lemmas 5.7.24, 5.7.26, and 5.6.18 iv).

**Definition 5.7.23.** We say that a binary relation \( R \) is \( x \)-special if and only if

i) \( \text{NAT}(x) \);

ii) \( \text{dom}(R) = \{y: y \leq x\} \);

iii) \( (\forall y)(R(0, y) \leftrightarrow y \text{ is a } <# \text{ minimal element}) \);

iv) \( (\forall y \leq x)(\forall w)(R(y+1, w) \leftrightarrow (\exists z)(R(y, z) \land w \text{ is an immediate successor of } z \text{ in } <#))) \).

We claim that for all \( x \) with \( \text{NAT}(x) \), there exists an \( x \)-special \( R \). This is proved by induction, which is supported by Lemma 5.6.18 iv), vi), vii), and Lemma 5.7.5. The basis case \( x = 0 \) is immediate.

For the induction case, let \( R \) be \( x \)-special. By Lemma 5.7.5, define

\[
S(y, w) \leftrightarrow R(y, w) \lor (y = x+1 \land (\exists z)(R(x, z) \land w \text{ is an immediate successor of } z \text{ in } <#))).
\]

uniquely up to \( \equiv' \). We claim that \( S \) is \( x+1 \)-special. It is clear that \( \text{dom}(S) = \{y: y \leq x+1\} \) since \( \text{dom}(R) = \{y: y \leq x\} \).
and we can find immediate successors in <#. Also the conditions

\[(\forall y)(S(0,y) \leftrightarrow y \text{ is a } <\# \text{ minimal element}).\]
\[(\forall y \leq x)(\forall w)(S(y+1,w) \leftrightarrow (\exists z)(R(y,z) \land w \text{ is an immediate successor of } z \text{ in } <\#)).\]

are inherited from R. To see that

\[(\forall w)(S(x+1,w) \leftrightarrow (\exists z)(S(x,z) \land w \text{ is an immediate successor of } z \text{ in } <\#))\]

we need to know that \(\{z: R(x,z)\}\) forms an equivalence class under \(=\). This is proved by induction on \(x\) from 0 through \(x\).

We have thus shown that there exists an \(x\)-special \(R\) for all \(x\) with \(\text{NAT}(x)\). Another induction on \(\text{NAT}\) shows that

1) \(\text{NAT}(x) \land \text{NAT}(y) \land x \leq y \land R \text{ is } x\)-special \land
\(S \text{ is } y\)-special \land z \leq x \rightarrow \)
\(R(z,w) \leftrightarrow S(z,w).\)

We also claim that

\(\text{NAT}(x) \rightarrow \)

there exists an \(x\)-special \(R < d_2\).

To see this, let \(\text{NAT}(x)\). By Lemma 5.6.18 iii), let \(n > 1\) be so large that

\((\exists y < d_n)(y \text{ is } x\)-special).\)

By Lemma 5.6.18 vi), \(x < d_1\). Hence by Lemma 5.6.18 v),

\((\exists y < d_2)(y \text{ is } x\)-special).\)

Because of this \(d_2\) bound, we can apply Lemma 5.7.5 to form a union \(\text{RNAT}\) of the \(x\)-special relations with \(\text{NAT}(x)\), uniquely up to \(=\). Claims i)-iii) are easily verified using 1). Thus we have

\((\exists R)(R \text{ is } \text{an RNAT} \land R \text{ obeys clauses i)-iii})).\)

Hence by Lemma 5.6.18 v),

\((\exists R < d_2)(R \text{ is } \text{an RNAT} \land R \text{ obeys clauses i)-iii})).\)
\((\exists R)(R \text{ obeys clauses i)-iv})\).

The remaining claims can be proved from properties i)-iii) by induction. QED

**DEFINITION 5.7.24.** We fix the RNAT of Lemma 5.7.27, which is unique up to 

The limit point provided by the next Lemma will be used to interpret \(\omega\).

**LEMMA 5.7.28.** There is a \(<#\) least limit point of \(<#\). I.e., there exists \(x\) such that

i) \((\exists y)(y <# x)\);

ii) \((\forall y <# x)(\exists z <# x)(y <# z)\);

iii) for all \(x^*\) with properties i),ii), \(x \leq # x^*\).

All \(<#\) least limit points of \(<#\) are =#, and \(< d_2\).

Proof: We say that \(z\) is an \(\omega\) if and only if \(z\) is a \(<#\) least limit point of \(<#\); i.e., \(z\) obeys i)-iii).

By an obvious induction, if NAT\((x)\) then \(\{z : (\exists y \leq x) (RNAT(y,z))\}\) forms an initial segment of \(<#\). Therefore rng\((RNAT)\) forms an initial segment of \(<#\). Since RNAT \(< d_2\),

rng\((RNAT) \subseteq [0,d_2]\). According to Lemma 5.7.24, let \(z\) be \(<#\) least such that \((\forall x \in \text{rng}(RNAT))(x <# z)\).

It is clear that \(z\) obeys claims i),ii). Suppose \(x^*\) has properties i),ii). By an obvious induction, we see that \((\forall y \in \text{rng}(RNAT))(y <# x^*)\). Hence \(z \leq # x^*\). Thus we have verified claim iii) for \(z\). I.e., \(z\) is an \(\omega\).

Suppose \(z, z^*\) are \(\omega\)'s. By iii), \(z \leq # z^*,\) \(z^* \leq # z\). Hence \(z = # z^*\).

By Lemma 5.6.18 iii), let \(n > 1\) be such that

"there exists an \(\omega < d_n\)."

Hence By Lemma 5.6.18 v),

"there exists an \(\omega < d_2\)."

Finally, we establish that every \(\omega\) is \(< d_2\). Suppose

"there exists an \(\omega > d_2\)."
By Lemma 5.6.18 v),

"there exists an \( \omega > d_3 \)."

Hence the \( \omega \)'s form an interval, with an element \( < d_2 \) and an element \( > d_3 \). Hence \( d_2 = d_3 \). This contradicts Lemma 5.7.26.

QED

We are now prepared to define the system \( M^\).

**DEFINITION 5.7.25.** \( M^ = (C,<,0,1,+,-,\cdot,↑,log,\omega,c_1,c_2,...,Y_1,Y_2,...), \) where the following components are defined below.

i) \( (C,<) \) is a linear ordering;
ii) \( c_1,c_2,... \) are elements of \( C \);
iii) for \( k \geq 1, Y_k \) is a set of \( k \)-ary relations on \( C \);
iv) \( 0,1,\omega \) are elements of \( C \);
v) \( +,-,\cdot \) are binary functions from \( C \) into \( C \);
vi) \( ↑,log \) are unary functions from \( C \) into \( C \).

**DEFINITION 5.7.26.** For \( x ∈ D \), we write \([x]\) for the equivalence class of \( x \) under \( =^\). Recall from Lemma 5.7.26 that each \([x]\) is a nonempty interval in \((D,<)\).

**DEFINITION 5.7.27.** We define \( C = \{[x]; x ∈ D\} \). We define \([x]<[y] ↔ x <^# y\). For all \( n ≥ 1 \), we define \( c_n = [d_{n+1}] \).

**DEFINITION 5.7.28.** Let \( k ≥ 1 \). We define \( Y_k \) to be the set of all \( k \)-ary relations \( R \) on \( C \), where there exists a \( k \)-ary relation \( S \) on \( D \), internal to \( M^\), (i.e., given by a point in \( D \)), such that for all \( x_1,...,x_k ∈ C \),

\[
R(x_1,...,x_k) ↔ (\exists y_1,...,y_k ∈ D)(y_1 ∈ x_1 ∧ ... ∧ y_k ∈ x_k ∧ S(y_1,...,y_k)).
\]

Since \( k \)-ary relations \( S \) on \( D \) are required to be bounded in \( D \), by Lemma 5.7.26 every \( R ∈ Y_k \) is bounded in \( C \).

**DEFINITION 5.7.29.** By Lemma 5.7.28, we define the \( ω \) of \( M^\) to be \([z]\), where \( z \) is an \( ω \) of \( M^\), as defined in the first line of the proof of Lemma 5.7.28.

**DEFINITION 5.7.30.** Define the following function \( f \) externally. For each \( x ∈ D \) such that \( NAT(x) \), let \( f(x) = \{y: RNAT(x,y)\} \). Note that by Lemma 5.7.27, \( f(x) ∈ C \). Note that
the relation \( y \in f(x) \) is internal to \( M^\# \). Also by Lemma 5.7.28 and an internal induction argument, \( f \) is one-one.

**DEFINITION 5.7.31.** We define 0 to be \( f(0) = [0] \), and 1 to be \( f(1) \).

**DEFINITION 5.7.32.** For \( x, y \) such that \( \text{NAT}(x), \text{NAT}(y) \), we define

\[
\begin{align*}
    f(x) + f(y) &= f(x+y), \\
    f(x) - f(y) &= f(x-y), \\
    f(x) \cdot f(y) &= f(x \cdot y), \\
    f(x) \uparrow &= f(x \uparrow), \\
    \log(f(x)) &= f(\log(x)).
\end{align*}
\]

Here the operations on the left side are in \( M^\wedge \), and the operations on the right side are in \( M^\# \). Note that the above definitions of \(+, -, \cdot, \log\) on \( \text{rng}(f) \) are internal to \( M^\# \).

**DEFINITION 5.7.33.** Let \( u, v \in C \), where \( \neg (u, v \in \text{rng}(f)) \). We define

\[
    u + v = u - v = u \cdot v = u \uparrow = \log(u) = [0].
\]

We now define the language \( L^\wedge \) suitable for \( M^\wedge \), without the \( c \)'s.

**DEFINITION 5.7.34.** \( L^\wedge \) is based on the following primitives.

i) The binary relation symbol \(<\);
ii) The constant symbols \( 0, 1, \omega \);
iii) The unary function symbols \( \uparrow, \log \);
iv) The binary function symbols \(+, -, \cdot\);
v) The first order variables \( v_n \), \( n \geq 1 \);
vii) The second order variables \( B^n_m \), \( n, m \geq 1 \);

In addition, we use \( \forall, \exists, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, = \). Commas and parentheses are also used. "B" indicates "bounded set".

**DEFINITION 5.7.35.** The first order terms of \( L^\wedge \) are inductively defined as follows.

i) The first order variables \( v_n \), \( n \geq 1 \) are first order terms of \( L^\wedge \);
ii) The constant symbols \( 0, 1, \omega \) are first order terms of \( L^\wedge \);
iii) If \( s, t \) are first order terms of \( L^\wedge \) then \( s+t, s-t, s \cdot t, t \uparrow, \log(t) \) are first order terms of \( L^\wedge \).
DEFINITION 5.7.36. The atomic formulas of \( L^\) are of the form

\[
\begin{align*}
  s &= t \\
  s &< t \\
  B^n_m(t_1, \ldots, t_n)
\end{align*}
\]

where \( s, t, t_1, \ldots, t_n \) are first order terms and \( n \geq 1 \). The formulas of \( L^\) are built up from the atomic formulas of \( L^\) in the usual way using the connectives and quantifiers.

Note that there is no epsilon relation in \( L^\).

The first order quantifiers range over \( C \). The second order quantifiers \( B^n_k \) range over \( Y_n \).

LEMMA 5.7.29. Let \( k \geq 1 \) and \( R \subseteq C^k \) be \( M^\) definable (with first and second order parameters allowed). Then \( \{(x_1, \ldots, x_k) : R([x_1], \ldots, [x_k])\} \) is \( M^\#$ definable (with parameters allowed). If \( R \) is \( M^\) definable without parameters, then \( \{(x_1, \ldots, x_k) : R([x_1], \ldots, [x_k])\} \) is \( M^\#$ definable without parameters.

Proof: The construction of \( M^\) takes place in \( M^\# \), where equality in \( M^\) is given by the equivalence relation \( =^\# \) in \( M^\# \). Note that \( =^\# \) is defined in \( M^\# \) without parameters. The \( <, 0, 1, \omega \) of \( M^\) are also defined without parameters.

Let \( k \geq 1 \). The relations in \( Y_k \) are each coded by arbitrary internal \( k \)-ary relations \( R \) in \( M^\# \), where the application relation “the relation coded by \( R \) holds at points \( x_1, \ldots, x_k \)” is defined in \( M^\# \) without parameters.

Using these considerations, it is straightforward to convert \( M^\) definitions to \( M^\# \) definitions. QED

LEMMA 5.7.30. There exists a structure \( M^\) = \((C, <, 0, 1, +, -, \cdot, \uparrow, \log, \omega, c_1, c_2, \ldots, Y_1, Y_2, \ldots)\) such that the following holds.

i) \((C, <)\) is a linear ordering;

ii) \( \omega \) is the least limit point of \((C, <)\);

iii) \( \{(x : x < \omega), <, 0, 1, +, -, \cdot, \uparrow, \log\} \) satisfies TR(\( \Pi^0_1, L \));

iv) For all \( x, y \in C \), \( \neg(x < \omega \land y < \omega) \rightarrow x + y = x \cdot y = x - y = x \uparrow = \log(x) = 0; \)

v) The \( c_n \), \( n \geq 1 \), form a strictly increasing sequence of elements of \( C \), all > \( \omega \), with no upper bound in \( C \).
vi) For all $k \geq 1$, $Y_k$ is a set of $k$-ary relations on $C$ whose field is bounded above;

vii) Let $k \geq 1$, and $\varphi$ be a formula of $L^\wedge$ in which the $k$-ary second order variable $B^k_n$ is not free, and the variables $B^m_r$ range over $Y_r$. Then $(\exists B^k_n \in Y_k)(\forall x_1, \ldots, x_k)(B^k_n(x_1, \ldots, x_k) \iff (x_1, \ldots, x_k \leq y \wedge \varphi))$;

viii) Every nonempty $M^\wedge$ definable subset of $C$ has a $<$ least element;

ix) Let $r \geq 1$ and $\varphi(v_1, \ldots, v_{2r})$ be a formula of $L^\wedge$. Let $1 \leq i_1, \ldots, i_r$, where $(i_1, \ldots, i_r)$ and $(i_{r+1}, \ldots, i_{2r})$ have the same order type and the same min. Let $y_1, \ldots, y_r \in C$, $y_1, \ldots, y_r \leq \min(c_{i_1}, \ldots, c_{i_r})$. Then $\varphi(c_{i_1}, \ldots, c_{i_r}, y_1, \ldots, y_r) \iff \varphi(c_{i_{r+1}}, \ldots, c_{i_{2r}}, y_1, \ldots, y_r)$.

Proof: We show that the $M^\wedge$ we have constructed obeys these properties. Claim i) is by construction, since $<$ is irreflexive, transitive, and has trichotomy. Claim ii) is by the definition of $\omega$ (see Definition 5.7.29).

For claim iii), note that the $f$ used in the construction of $M^\wedge$ defines an isomorphism from the $\langle \{x: \text{NAT}(x)\}, 0, 1, +, -, \cdot, \uparrow, \log \rangle$ of $M#$ onto the $\langle \{x: x < \omega\}, <, 0, 1, +, -, \cdot, \uparrow, \log \rangle$ of $M^\wedge$. Now apply Lemma 5.6.18 viii).

Claim iv) is by construction.

For claim v), for all $n \geq 1$, $c_n = \lceil d_{n+1} \rceil$. By Lemma 5.7.26, the $c_n$'s are strictly increasing. Let $\lceil x \rceil \in C$. By Lemma 5.6.18 iii), let $x < d_{m+1}$, in $M#$. By Lemma 5.7.17, $\neg(d_{m+1} < \# x)$. Therefore $x \# d_{m+1}$. Hence $\lceil x \rceil \leq [d_{m+1}] = c_m$. Hence the $c_n$'s have no upper bound in $C$. By Lemma 5.7.27, any $\omega$ of $M#$ is $< \# d_2$ in $M#$. Hence $\omega < c_1$ in $M^\wedge$.

Claim vi) is by construction. This uses that there is no $< \#$ greatest point in $M#$ (Lemma 5.7.26).

For claim vii), it suffices to show that every $M^\wedge$ definable relation $R$ on $C$ whose field is bounded above ($\leq$ on $C$) lies in $Y_k$. By Lemma 5.7.29, the $k$-ary relation $S$ on $D$ given by

$$S(y_1, \ldots, y_k) \iff R([y_1], \ldots, [y_k])$$

is $M#$ definable. Since the field of $R$ is bounded above ($\leq$ on $C$), the field of $S$ is bounded above ($\leq$ on $D$). This uses that $<$ on $C$ has no greatest element (Lemma 5.7.26). Hence we can take $S$ to be internal to $M#$; i.e., given by a point in $D$. Therefore $R \in Y_k$. 

For claim viii), let $R$ be a nonempty $M^\dagger$ definable subset of $C$. By Lemma 5.7.29, $S = \{y: [y] \in R\}$ is nonempty and $M^\#$ definable. By Lemma 5.7.24, let $y$ be a $<$\# least element of $S$.

We claim that in $M^\dagger$, $[y]$ is the $<$ least element of $R$. To see this, let $[z] \in R$, $[z] < [y]$. Then $z <\# y$ and $z \in S$, which contradicts the choice of $y$.

For claim ix), let $\varphi(x_1, \ldots, x_{2r}, i_1, \ldots, i_{2r}, y_1, \ldots, y_r)$ be as given. Let $i = \min(i_1, \ldots, i_r)$. Since $y_1, \ldots, y_r \leq c_1 = [d_{i+1}]$, every element of the equivalence classes $y_1, \ldots, y_r$ is $\leq$\# $d_{i+1}$. Hence we can write $y_1 = [z_1], \ldots, y_r = [z_r]$, where $z_1, \ldots, z_r \leq d_{i+1}$.

By Lemma 5.7.29, the $2r$-ary relation $S$ on $D$ given by

$$
S(w_1, \ldots, w_{2r}) \iff 
\varphi([w_1], \ldots, [w_{2r}]) \text{ holds in } M^\dagger
$$

is definable in $M^\#$ without parameters.

Note that $\min(i_1+1, \ldots, i_{2r}+1) = i+1$. Hence by Lemma 5.6.18 v), we have

$$
S(d_{i_1+1}, \ldots, d_{i_r+1}, z_1, \ldots, z_r) \iff 
S(d_{i_r+1+1}, \ldots, d_{i_{2r}+1}, z_1, \ldots, z_r).
$$

Hence in $M^\dagger$,

$$
\varphi(c_{i_1}, \ldots, c_{i_r}, [z_1], \ldots, [z_r]) \iff 
\varphi(c_{i_r+1}, \ldots, c_{i_{2r}+1}, [z_1], \ldots, [z_r]).
$$

$$
\varphi(c_{i_1}, \ldots, c_{i_r}, y_1, \ldots, y_r) \iff 
\varphi(c_{i_r+1}, \ldots, c_{i_{2r}}, y_1, \ldots, y_r).
$$

QED