3.4. AAAA.

Recall the AA table from section 3.3.

AA

1. $A \cup f_A \subseteq A \cup g_A$. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2. $A \cup f_A \subseteq A \cup g_B$. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3. $A \cup f_A \subseteq A \cup g_C$. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4. $B \cup f_A \subseteq A \cup g_A$. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. $B \cup f_A \subseteq A \cup g_B$. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. $B \cup f_A \subseteq A \cup g_C$. ¬INF. AL. ¬ALF. ¬FIN. NON.
7. $C \cup f_A \subseteq A \cup g_A$. ¬INF. AL. ¬ALF. ¬FIN. NON.
8. $C \cup f_A \subseteq A \cup g_B$. ¬INF. AL. ¬ALF. ¬FIN. NON.
9. $C \cup f_A \subseteq A \cup g_C$. ¬INF. AL. ¬ALF. ¬FIN. NON.

It is clear that there is no reason to further consider clauses 1-3 from the AA table, as all of our five proposition attributes already come out false. So we instead work with the following reduced AA table. Note that we have renumbered the clauses.

REDUCED AA

1. $B \cup f_A \subseteq A \cup g_A$. ¬INF. AL. ¬ALF. ¬FIN. NON.
2. $B \cup f_A \subseteq A \cup g_B$. ¬INF. AL. ¬ALF. ¬FIN. NON.
3. $B \cup f_A \subseteq A \cup g_C$. ¬INF. AL. ¬ALF. ¬FIN. NON.
4. $C \cup f_A \subseteq A \cup g_A$. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. $C \cup f_A \subseteq A \cup g_B$. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. $C \cup f_A \subseteq A \cup g_C$. ¬INF. AL. ¬ALF. ¬FIN. NON.

We need only consider ordered pairs of these clauses i,j, where $i < j$.

1,2. $B \cup f_A \subseteq A \cup g_A$, $B \cup f_A \subseteq A \cup g_B$.
1,3. $B \cup f_A \subseteq A \cup g_A$, $B \cup f_A \subseteq A \cup g_C$.
1,4. $B \cup f_A \subseteq A \cup g_A$, $C \cup f_A \subseteq A \cup g_A$.
1,5. $B \cup f_A \subseteq A \cup g_A$, $C \cup f_A \subseteq A \cup g_B$.
1,6. $B \cup f_A \subseteq A \cup g_A$, $C \cup f_A \subseteq A \cup g_C$.
2,3. $B \cup f_A \subseteq A \cup g_B$, $B \cup f_A \subseteq A \cup g_C$.
2,4. $B \cup f_A \subseteq A \cup g_B$, $C \cup f_A \subseteq A \cup g_A$. Equivalent to 1,6.
2,5. $B \cup f_A \subseteq A \cup g_B$, $C \cup f_A \subseteq A \cup g_B$.
2,6. $B \cup f_A \subseteq A \cup g_B$, $C \cup f_A \subseteq A \cup g_C$.
3,4. $B \cup f_A \subseteq A \cup g_C$, $C \cup f_A \subseteq A \cup g_A$. Equivalent to 1,5.
3,5. $B \cup f_A \subseteq A \cup g_C$, $C \cup f_A \subseteq A \cup g_B$. 

3, 6. \( B \cup fA \subseteq A \cup gC, C \cup fA \subseteq A \cup gC \). Equivalent to 2, 5.

4, 5. \( C \cup fA \subseteq A \cup gA, C \cup fA \subseteq A \cup gB \). Equivalent to 1, 3.

4, 6. \( C \cup fA \subseteq A \cup gA, C \cup fA \subseteq A \cup gC \). Equivalent to 1, 2.

5, 6. \( C \cup fA \subseteq A \cup gB, C \cup fA \subseteq A \cup gC \). Equivalent to 2, 3.

Thus we need only examine

**REDUCED AAAA**

1, 2. \( B \cup fA \subseteq A \cup gA, B \cup fA \subseteq A \cup gB \). ¬INF. ¬AL.

¬ALF. ¬FIN. ¬NON.

1, 3. \( B \cup fA \subseteq A \cup gA, B \cup fA \subseteq A \cup gC \). ¬INF. AL. ¬ALF.

¬FIN. NON.

1, 4. \( B \cup fA \subseteq A \cup gA, C \cup fA \subseteq A \cup gA \). ¬INF. AL. ¬ALF.

¬FIN. NON.

1, 5. \( B \cup fA \subseteq A \cup gA, C \cup fA \subseteq A \cup gB \). ¬INF. AL.

¬ALF. ¬FIN. ¬NON.

1, 6. \( B \cup fA \subseteq A \cup gA, C \cup fA \subseteq A \cup gC \). ¬INF. AL.

¬ALF. ¬FIN. ¬NON.

2, 3. \( B \cup fA \subseteq A \cup gB, B \cup fA \subseteq A \cup gC \). ¬INF. AL. ¬ALF.

¬FIN. NON.

2, 5. \( B \cup fA \subseteq A \cup gB, C \cup fA \subseteq A \cup gB \). ¬INF. AL. ¬ALF.

¬FIN. NON.

2, 6. \( B \cup fA \subseteq A \cup gB, C \cup fA \subseteq A \cup gC \). ¬INF. AL. ¬ALF.

¬FIN. NON.

3, 5. \( B \cup fA \subseteq A \cup gC, C \cup fA \subseteq A \cup gB \). ¬INF. AL. ¬ALF.

¬FIN. NON.

Note that we have used an entirely different method for compiling the ordered pairs to be analyzed than the purely syntactic method used in section 3.1 to compile the master list for AAAA that is used in the Annotated Table, section 3.14. Here we take full advantage of the fact that ¬NON implies ¬INF, ¬AL, ¬ALF, ¬FIN. The result is that on the master list for AAAA, there are 20 entries, whereas on the above Reduced AAAA list, there are only 9 entries.

The same considerations apply in sections 3.5 – 3.13, where the number of ordered pairs actually requiring analysis is considerably smaller than the number of ordered pairs in the relevant part of the Annotated Table.
By the reduced AA table, we see that all of these pairs must have \(-\)INF, \(-\)ALF, \(-\)FIN. It remains to determine the status of AL and NON.

In the next Lemma, we use this method of substitution: Suppose \(\alpha, \beta\) are pairs of clauses, where \(\beta\) is the result of substituting one letter by another letter in \(\alpha\). Then any of our five attributes that holds of \(\beta\) also holds of \(\alpha\). As a consequence, if the negation of any of our five attributes holds of \(\alpha\) then that negation also holds of \(\beta\).

**Lemma 3.4.1.** 1,3, 1,4 have AL.

Proof: From the reduced AA table, \(B \cup fA \subseteq A \cup gA\) has AL. In the cited ordered pairs, replace C by A, and C by B, respectively. QED

The following pertains to 1,2, 1,5, 1,6.

**Lemma 3.4.2.** \(fX \subseteq A \cup gA\), \(fA \subseteq A \cup gY\), \(Y \cap fA = \emptyset\) has \(-\)NON.

Proof: Let \(f\) be as given by Lemma 3.2.1. Let \(g \in \text{ELG}\) be defined by \(g(n) = 2n+1\). Let \(fX \subseteq A \cup gA\), \(fA \subseteq A \cup gY\), \(Y \cap fA = \emptyset\), where \(A,B,C\) are nonempty.

Let \(n \in fA \cap 2N\). Then \(n \in A\). Hence \(fA \cap 2N \subseteq A\). By Lemma 3.2.1, \(fA\) is cofinite. Since \(Y \cap fA = \emptyset\), \(Y\) is finite. Hence \(A\) is cofinite. This contradicts \(A \cap gA = \emptyset\). QED

The following pertains to 2,3, 2,5, 2,6.

**Lemma 3.4.3.** \(B \cup fA \subseteq A \cup gB\), \(X \cup fA \subseteq A \cup gY\) has AL, provided \(X,Y \in \{B,C\}\).

Proof: From the reduced AA table, \(B \cup fA \subseteq A \cup gB\) has AL. In the cited ordered pairs, replace C by B. QED

The following pertains to 3,5.

**Lemma 3.4.4.** \(B \cup fA \subseteq A \cup gC\), \(C \cup fA \subseteq A \cup gX\) has AL, provided \(X \in \{B,C\}\).

Proof: From the reduced AA table, \(B \cup fA \subseteq A \cup gB\) has AL. In the cited ordered pair, replace C by B. QED