3.12. ABBC.

Recall the following reduced table for AB from section 3.5.

REduced AB

1. A U. fA C B U. gA. INF. AL. ALF. FIN. NON.
2. A U. fA C B U. gB. INF. AL. ALF. FIN. NON.
3. A U. fA C B U. gC. INF. AL. ALF. FIN. NON.
4. C U. fA C B U. gA. INF. AL. ALF. FIN. NON.
5. C U. fA C B U. gB. INF. AL. ALF. FIN. NON.
6. C U. fA C B U. gC. INF. AL. ALF. FIN. NON.

The reduced table for BC is obtained from the reduced table for AB via the permutation that sends A to B, B to C, and C to A. We use 1'-6' to avoid confusion.

REduced BC

1'. B U. fB C C U. gB. INF. AL. ALF. FIN. NON.
2'. B U. fB C C U. gC. INF. AL. ALF. FIN. NON.
3'. B U. fB C C U. gA. INF. AL. ALF. FIN. NON.
4'. A U. fB C C U. gB. INF. AL. ALF. FIN. NON.
5'. A U. fB C C U. gC. INF. AL. ALF. FIN. NON.
6'. A U. fB C C U. gA. INF. AL. ALF. FIN. NON.

This results in 36 ordered pairs, which we divide into six cases. We begin with two Lemmas.

We will determine the status of all attributes INF, AL, ALF, FIN, NON, for all ordered pairs.

LEmma 3.12.1. C U. fX C B U. gY, Z U. fB C C U. gW has ¬INF, ¬FIN.

Proof: Let f be as given by Lemma 3.2.1. Let g ∈ ELG be given by g(n) = 2n+1. Let C U. fX C B U. gY, Z U. fB C C U. gW, where A, B, C are nonempty.

Clearly fB ∩ 2N C C. By C C B U gY, we have fB ∩ 2N C B. Hence by Lemma 3.2.1, fB is cofinite. Hence B is infinite. This establishes that ¬FIN. Also Z is finite. This establishes that ¬INF. QED

LEmma 3.12.2. C U. fX C B U. gY, Z U. fB C C U. gW, B ∩ fB = Ø has ¬NON.
Proof: We can continue the proof of Lemma 3.12.1. Using \( fB \) is cofinite and \( B \) is finite, we obtain an immediate contradiction from \( B \cap fB = \emptyset \). QED

We use Lemmas 3.12.1 and 3.12.2 in cases 5,6 below.

part 1. \( A \cup fA \subseteq B \cup gA \).

1,1'. \( A \cup fA \subseteq B \cup gA \), \( B \cup gB \subseteq C \cup gC \).

1,2'. \( A \cup fA \subseteq B \cup gA \), \( B \cup fB \subseteq C \cup gC \).

1,3'. \( A \cup fA \subseteq B \cup gA \), \( B \cup fB \subseteq C \cup gA \).

1,4'. \( A \cup fA \subseteq B \cup gA \), \( A \cup fB \subseteq C \cup gC \).

1,5'. \( A \cup fA \subseteq B \cup gA \), \( A \cup fB \subseteq C \cup gC \).

1,6'. \( A \cup fA \subseteq B \cup gA \), \( A \cup fB \subseteq C \cup gA \).

The following pertains to 1,4', 1,6'.

LEMMA 3.12.3. \( A \cup fA \subseteq B \cup gA \), \( A \cup fB \subseteq C \cup gC \) has INF, ALF provided \( X \in \{A,B\} \), even for EVSD.

Proof: Let \( f,g \in EVSD \). Let \( n \) be sufficiently large. By Lemma 3.2.5, let \( A \subseteq [n,\infty) \) be infinite, where \( A \) is disjoint from \( f(A \cup fA) \cup g(A \cup fA) \). Let \( B = (A \cup fA) \setminus gA \), and \( C = (A \cup fB) \setminus gX \).

Clearly \( A \cap fA = B \cap gA = A \cap fB = C \cap gX = A \cap gA = A \cap gB = \emptyset \). Hence \( A \subseteq B \) and \( A \subseteq C \). Also \( fA \subseteq B \cup gA \) and \( fB \subseteq C \cup gX \). This establishes INF.

We can repeat the argument where \( A \) is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 1,5'.

LEMMA 3.12.4. \( A \cup fA \subseteq B \cup gA \), \( A \cup fB \subseteq C \cup gC \) has INF, ALF, even for EVSD.

Proof: Let \( f,g \in EVSD \). Let \( n \) be sufficiently large. By Lemma 3.2.5, let \( A \subseteq [n,\infty) \) be infinite, where \( A \) is disjoint from \( f(A \cup fA) \cup g(A \cup fA) \cup g(A \cup f(A \cup fA)) \). Let \( B = (A \cup fA) \setminus gA \), and \( C = (A \cup fB) \setminus gX \).

Clearly \( A \cap fA = B \cap gA = A \cap fB = C \cap gX = A \cap gA = A \cap gB = \emptyset \). Hence \( A \subseteq B \) and \( A \subseteq C \). Also \( fA \subseteq B \cup gA \) and \( fB \subseteq C \cup gX \). This establishes INF.
fA)\gA. By Lemma 3.3.3, let C be unique such that C \subseteq A \cup \ fB \subseteq C \cup \ gC.

Clearly A \cap fA = B \cap gA = A \cap fB = C \cap gC = A \cap gA = A \cap gC = \emptyset. Hence A \subseteq B and A \subseteq C. Also fA \subseteq B \cup gA and fB \subseteq C \cup gC. This establishes INF.

We can repeat the proof where A is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 1,1', 1,2', 1,3'.

LEMMA 3.12.5. A \cup fA \subseteq B \cup gA, B \cap fB = \emptyset has \neg \text{NON}.

Proof: Define f,g \in ELG as follows. Let f(n) = 2n+2 and g(n) = 2n+1. Let A \cup fA \subseteq B \cup gA, B \cap fB = \emptyset, where A,B are nonempty.

Let n = \min(A). Then n \not\in gA, n \in B, 2n+2 \in fB, 2n+2 \in fA, 2n+2 \in B. This contradicts B \cap fB = \emptyset. QED

part 2. A \cup fA \subseteq B \cup gB.

2,1'. A \cup fA \subseteq B \cup gB, B \cup fB \subseteq C \cup gB. \neg \text{INF}. \neg \text{AL}.
\neg \text{ALF}. \text{FIN}. \text{NON}.
2,2'. A \cup fA \subseteq B \cup gB, B \cup fB \subseteq C \cup gC. \neg \text{INF}. \neg \text{AL}.
\neg \text{ALF}. \text{FIN}. \text{NON}.
2,3'. A \cup fA \subseteq B \cup gB, B \cup fB \subseteq C \cup gA. \neg \text{INF}. \neg \text{AL}.
\neg \text{ALF}. \text{FIN}. \text{NON}.
2,4'. A \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gB. \text{INF}. \text{AL}. \text{ALF}.
\text{FIN}. \text{NON}.
2,5'. A \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gC. \text{INF}. \text{AL}. \text{ALF}.
\text{FIN}. \text{NON}.
2,6'. A \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gA. \text{INF}. \text{AL}. \text{ALF}.
\text{FIN}. \text{NON}.

The following pertains to 2,4', 2,6'.

LEMMA 3.12.6. A \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gX has \text{INF}, \text{ALF}, provided X \in \{A,B\}, even for EVSD.

Proof: Let f,g \in EVSD. Let n be sufficiently large. By Lemma 3.2.5, let A \subseteq [n,\infty) be infinite, where A is disjoint from f(A \cup fA) \cup g(A \cup fA). By Lemma 3.3.3, let B be unique such that B \subseteq A \cup fA \subseteq B \cup gb. Let C = (A \cup fB)\setminus gX.
Clearly $A \cap fA = B \cap gB = A \cap fB = C \cap gC = A \cap gB = A \cap gX$. Hence $A \subseteq B$ and $A \subseteq C$. Also $fA \subseteq B \cup gB$ and $fB \subseteq C \cup gX$. This establishes INF.

We can repeat the argument where $A$ is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 2,5'.

LEMMA 3.12.7. $A \cup fA \subseteq B \cup gB$, $A \cup fB \subseteq C \cup gC$ has INF, ALF, even for EVSD.

Proof: Let $f,g \in$ EVSD. Let $n$ be sufficiently large. By Lemma 3.2.5, let $A \subseteq [n,\infty)$ be infinite, where $A$ is disjoint from $f(A \cup fA) \cup g(A \cup fA) \cup g(A \cup f(A \cup fA))$. By Lemma 3.3.3, let $B$ be unique such that $B \subseteq A \cup fA \subseteq B \cup gB$. By Lemma 3.3.3, let $C$ be unique such that $C \subseteq A \cup fB \subseteq C \cup gC$.

Clearly $A \cap fA = B \cap gB = A \cap fB = C \cap gC = A \cap gB = A \cap gX$. Hence $A \subseteq B$ and $A \subseteq C$. Also $fA \subseteq B \cup gB$ and $fB \subseteq C \cup gC$. This establishes INF.

We can repeat the argument where $A$ is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 2,1', 2,3'.

LEMMA 3.12.8. $A \cup fA \subseteq B \cup gB$, $B \cup fB \subseteq C \cup gX$ has FIN, provided $X \in \{A,B\}$.

Proof: Let $f,g \in$ ELG. We claim that there exists arbitrarily large $n$ such that $f(n,\ldots,n) \neq f(g(n,\ldots,n),\ldots,g(n,\ldots,n))$. Suppose this is false. I.e., let $r$ be such that for all $n \geq r$, $f(n,\ldots,n) = f(g(n,\ldots,n),\ldots,g(n,\ldots,n))$. We can assume that $r$ is chosen so that $f,g$ is strictly dominating on $[r,\infty)$.

Define $t_0 = r$, $t_{i+1} = g(t_i,\ldots,t_i)$. An obvious induction shows that $r \leq t_0 < t_1 < \ldots$.

We now prove by induction that for all $i \geq 0$,

$$f(r,\ldots,r) = f(t_1,\ldots,t_1).$$

Obviously this is true for $i = 0$. Suppose this is true for a given $i \geq 0$. Then
\[ f(r, \ldots, r) = f(t_1, \ldots, t_1). \]
\[ t_1 \geq r. \]
\[ f(t_1, \ldots, t_1) = f(g(t_1, \ldots, t_1), \ldots, g(t_1, \ldots, t_1)). \]
\[ f(r, \ldots, r) = f(t_{i+1}, \ldots, t_{i+1}). \]

However some \( t_1 \) is greater than \( f(r, \ldots, r) \), since the \( t \)'s are strictly increasing. This is a contradiction. The claim is now established.

Now let \( n \) be sufficiently large with the property that \( f(n, \ldots, n) \neq f(g(n, \ldots, n), \ldots, g(n, \ldots, n)) \). Let \( A = \{g(n, \ldots, n)\} \). Let \( B = \{n, f(g(n, \ldots, n), \ldots, g(n, \ldots, n))\} \). Let \( C = (B \cup fB) \setminus gX \).

Clearly \( A \cap fA = B \cap gB = B \cap fB = C \cap gX = \emptyset \). Also \( A \subseteq gB, fA \subseteq B, B \cup fB \subseteq C \cup gX \). In addition, \( n \notin gX, n \in B \), and so \( n \in C \). Hence \( A, B, C \) are nonempty finite sets. QED

The following pertains to 2,2'.

**Lemma 3.12.9.** \( A \cup fA \subseteq B \cup gB, B \cup fB \subseteq C \cup gC \) has FIN.

Proof: Let \( f, g \in ELG \). We define \( n, A, B \) exactly as in the proof of Lemma 3.12.8. By Lemma 3.3.3, let \( C \) be unique such that \( C \subseteq B \cup fB \subseteq C \cup gC \).

Clearly \( A \cap fA = B \cap gB = B \cap fB = C \cap gC = \emptyset \). Also \( A \subseteq gB, fA \subseteq B, B \cup fB \subseteq C \cup gC \). In addition, \( n \notin gC \), and so \( n \in C \). Hence \( A, B, C \) are nonempty finite sets. QED

The following pertains to 2,1', 2,2', 2,3'.

**Lemma 3.12.10.** \( fA \subseteq B \cup gX, B \cap fB = \emptyset \) has \( \neg \)AL.

Proof: Define \( f, g \in ELG \) as follows. For all \( n < m \), let \( f(n, n) = 2n+2, f(m, n) = f(n, m) = 4m+6, g(n) = 2n+1 \). Let \( fA \subseteq B \cup gX, B \cap fB = \emptyset \), where \( A, B, C \) have at least two elements. Let \( n < m \) be from \( A \). Then \( 2m+2,4m+6 \in fA, 2m+2,4m+6 \in B, 4m+6 \in fB \). This contradicts \( B \cap fB = \emptyset \). QED

part 3. \( A \cup fA \subseteq B \cup gC \).

3,1'. \( A \cup fA \subseteq B \cup gC, B \cup fB \subseteq C \cup gB \). \( \neg \)INF. \( \neg \)AL. \( \neg \)ALF. FIN. NON.
3, 2'. A U. fA ⊆ B U. gC, B U. fB ⊆ C U. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.

3, 3'. A U. fA ⊆ B U. gC, B U. fB ⊆ C U. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.

3, 4'. A U. fA ⊆ B U. gC, A U. fB ⊆ C U. gB. INF. AL. ALF. FIN. NON.

3, 5'. A U. fA ⊆ B U. gC, A U. fB ⊆ C U. gC. INF. AL. ALF. FIN. NON.

3, 6'. A U. fA ⊆ B U. gC, A U. fB ⊆ C U. gA. INF. AL. ALF. FIN. NON.

LEMMA 3.12.11. 3, 1' - 3, 3' have ¬AL.

Proof: By Lemma 3.12.10. QED

The following pertains to 3, 1', 3, 3'.

LEMMA 3.12.12. A U. fA ⊆ B U. gC, B U. fB ⊆ C U. gX has FIN, where X ∈ {A, B}.

Proof: Let f, g ∈ ELG. Let n be sufficiently large. Define A = {g(n, ..., n)}, B = {f(g(n, ..., n), ..., g(n, ..., n))}, C = (B U fB U {n}) \ gX.

Obviously A ∩ fA = B ∩ fB = C ∩ gX = ∅. Also n ∉ gX, n ∈ C. Hence A ⊆ gC and fA ⊆ B. Therefore A U fA ⊆ B U gC. Obviously B U fB ⊆ C U gX.

It remains to verify that B ∩ gC = ∅. Every element of C is either n or f(g(n, ..., n), ..., g(n, ..., n)) or the value of a term of depth ≤ 3 in f, g, n with f(g(n, ..., n), ..., g(n, ..., n)) as a subterm. Hence every element of gC is either g(n, ..., n) or the value of a term in f, g, n of depth ≤ 4 with f(g(n, ..., n), ..., g(n, ..., n)) as a proper subterm. Since n is sufficiently large, f(g(n, ..., n), ..., g(n, ..., n)) does not lie in gC. QED

The following pertains to 3, 2'.

LEMMA 3.12.13. A U. fA ⊆ B U. gC, B U. fB ⊆ C U. gC has FIN.

Proof: Let f, g ∈ ELG. Let n be sufficiently large. Define A = {g(n, ..., n)}, B = {f(g(n, ..., n), ..., g(n, ..., n))}. By Lemma 3.3.3, let C be unique such that C ⊆ B U fB U {n} ⊆ C U. gC.
Obviously $A \cap fA = B \cap fB = C \cap gC = \emptyset$. Also $n \notin gC$, $n \in C$. $A \subseteq gC$, and $fA \subseteq B$. Therefore $A \cup fA \subseteq B \cup gC$. In addition, $B \cup fB \subseteq C \cup gC$.

It remains to verify that $B \cap gC = \emptyset$. Argue exactly as in the proof of Lemma 3.12.12. QED

The following pertains to 3, 4', 3, 5', 3, 6'.

**Lemma 3.12.14.** $A \cup fA \subseteq B \cup gC$. $A \cup fB \subseteq C \cup gX$ has INF, ALF, even for EVSD.

**Proof:** Let $f, g \in EVSD$. Let $n$ be sufficiently large. By Lemma 3.2.5, let $A \subseteq [n, \infty)$ be infinite, where $A$ is disjoint from $f(A \cup fA) \cup g(A \cup f(A \cup fA))$. We inductively determine membership in $B, C$ for all elements of $[n, \infty)$. $B, C$ will have no elements $< n$.

Suppose membership in $B, C$ has been determined for all elements of $[n, k)$, $k \geq n$. We now determine membership in $B, C$ for $k$. If $k$ is already in $A \cup fA$ and $k$ is not yet in $gC$, put $k \in B$. If $k$ is already in $A \cup fB$ and $k$ is not yet in $gX$, put $k \in C$.

Clearly $B \subseteq A \cup fA$ and $C \subseteq A \cup fB \subseteq A \cup f(A \cup fA)$. Hence $A \cap fA = A \cap fB = C \cap gX = \emptyset$. Also $A \cup fA \subseteq B \cup gC$ and $A \cup fB \subseteq C \cup gX$. In addition, $A \cap fC \subseteq A \cap g(A \cup fB) \subseteq A \cap g(A \cup f(A \cup fA)) = \emptyset$, and so $A \cap gX = \emptyset$. Hence $A \subseteq B$, $A \subseteq C$.

This establishes INF.

We can instead use $A$ of any finite cardinality. We obtain finite $B, C$ with $A \subseteq B, C$. This establishes ALF. QED

part 4. $C \cup fA \subseteq B \cup gA$.

4, 1'. $C \cup fA \subseteq B \cup gA$, $B \cup fB \subseteq C \cup gB$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
4, 2'. $C \cup fA \subseteq B \cup gA$, $B \cup fB \subseteq C \cup gC$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
4, 3'. $C \cup fA \subseteq B \cup gA$, $B \cup fB \subseteq C \cup gA$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
4, 4'. $C \cup fA \subseteq B \cup gA$, $A \cup fB \subseteq C \cup gB$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
4, 5'. $C \cup fA \subseteq B \cup gA$, $A \cup fB \subseteq C \cup gC$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
4, 6'. $C \cup fA \subseteq B \cup gA$, $A \cup fB \subseteq C \cup gA$. ~INF. ~AL. ~ALF. ~FIN. ~NON.
The following pertains to 4,1', 4,2', 4,3'.

LEMMA 3.12.15. C \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq C \cup. gX has \neg NON.

Proof: Let f be as given by Lemma 3.2.1. Define g \in ELG by g(n) = 2n+1. Let C \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq C \cup. gX, where A,B,C are nonempty.

Let n \in fB \cap 2N. Then n \in C, n \in B. Hence fB \cap 2N \subseteq B. By Lemma 3.2.1, fB is cofinite. Hence B is infinite. This contradicts B \cap fB = \emptyset. QED

The following pertains to 4,4'.

LEMMA 3.12.16. C \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq C \cup. gB has \neg NON.

Proof: Define f,g \in ELG as follows. For all n < m, let f(2n,2n,2n) = f(2n+1,2n+1,2n+1) = 4n, f(n,m,m) = 2m, f(n,m,n) = 4m, f(m,n,n) = 8m, g(2n) = g(2n+1) = 4n+1. For all other triples a,b,c, let f(a,b,c) = 2|a,b,c|.

We claim that

f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(g(m),g(m),g(m)).

To see this, let m = 2r v m = 2r+1. Then

f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(4r,4r,4r) = 8r

and

f(g(m),g(m),g(m)) = f(4r+1,4r+1,4r+1) = 8r.

Now let C \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq C \cup. gB, where A,B,C are nonempty. Let n \in A. Then n \in C \cup. gB.

case 1. n \in C. Then n \in B \cup gA. First suppose n \in B. Then f(n,n,n) \in C \cup gB. Hence f(n,n,n) \in C. This contradicts C \cap fA = \emptyset.

Now suppose n \in gA. Let n = g(m), m \in A, m < n. Then 2n-2,4n-4,8n-8 \in fA, and so 2n-2,4n-4,8n-8 \in B, 8n-8 \in fB, 8n-8 \in C. This contradicts C \cap fA = \emptyset.
case 2. \( n \in gB. \) Let \( n = g(m), m \in B. \) Then \( f(m,m,m) \in fB, f(m,m,m) \in C. \) Hence \( f(m,m,m) \in B. \) Therefore
\[
f(f(m,m,m), f(m,m,m), f(m,m,m)) \in fB,
\]
\[
f(f(m,m,m), f(m,m,m), f(m,m,m)) \in C. \) Note that
\[
f(f(m,m,m), f(m,m,m), f(m,m,m)) = f(g(m), g(m), g(m)) = f(n, n, n) \in fA. \) This contradicts \( C \cap fA = \emptyset. \)

QED

The following pertains to 4,6'.

**LEMMA 3.12.17.** \( C U. fA \subseteq B U. gA, A U. fB \subseteq C U. gA \) has \( \neg \text{NON}. \)

Proof: Define \( f,g \) as in the proof of Lemma 3.12.16. Now let \( C U. fA \subseteq B U. gA, A U. fB \subseteq C U. gA, \) where \( A,B,C \) are nonempty. Let \( n \in A. \) Then \( n \in C U. gA. \)

**case 1.** \( n \in gA. \) Let \( n = g(m), m \in A, m < n. \) Then \( 2n-2, 4n-4, 8n-8 \in fA, 2n-2, 4n-4, 8n-8 \in B, 8n-8 \in fB, 8n-8 \in C. \) This contradicts \( C \cap fA = \emptyset. \)

**case 2.** \( n \in C. \) Then \( n \notin gA, n \in B, f(n,n,n) \in fB, f(n,n,n) \in C. \) Since \( f(n,n,n) \in fA, \) this contradicts \( C \cap fA = \emptyset. \)

QED

The following pertains to 4,5'.

**LEMMA 3.12.18.** \( C U. fA \subseteq B U. gA, A U. fB \subseteq C U. gC \) has \( \neg \text{NON}. \)

Proof: Define \( f,g \) as in the proof of Lemma 3.12.16. Now let \( C U. fA \subseteq B U. gA, A U. fB \subseteq C U. gC, \) where \( A,B,C \) are nonempty. Let \( n = \min(A). \) Then \( n \in C U. gC. \)

**case 1.** \( n \in C. \) By the choice of \( n, n \notin gA, n \in B. \) Hence \( f(n,n,n) \in fB, f(n,n,n) \in C. \) Since \( f(n,n,n) \in fA, \) this contradicts \( C \cap fA = \emptyset. \)

**case 2.** \( n \in gC. \) Let \( n = g(m), m \in C, m < n. \) Then \( m \in B U gA. \) By the choice of \( n, m \notin gA, m \in B. \) Hence \( f(m,m,m) \in fB, f(m,m,m) \in C, f(m,m,m) \in B U gA. \)

We claim that \( f(m,m,m) \notin gA. \) To see this, note that by quantitative considerations, \( f(m,m,m) \in gA \) implies that
there is an element of \( A \) that is \( \leq m < n \), which contradicts
the choice of \( n \).

Hence \( f(m,m,m) \in B \). Therefore
\[
\begin{align*}
f(f(m,m,m), f(m,m,m), f(m,m,m)) & \in fB. \\
f(f(m,m,m), f(m,m,m), f(m,m,m)) & \in C.
\end{align*}
\]

As in the proof of Lemma 3.12.16,
\[
f(f(m,m,m), f(m,m,m), f(m,m,m)) = f(g(m), g(m), g(m)) = f(n,n,n) \in fA.
\]

This contradicts \( C \cap fA = \emptyset \).

QED

part 5. \( C \cup fA \subseteq B \cup gB \).

5,1'. \( C \cup fA \subseteq B \cup gB \). B U. \( fB \subseteq C \cup U \). gB. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

5,2'. \( C \cup fA \subseteq B \cup gB \). B U. \( fB \subseteq C \cup U \). gC. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

5,3'. \( C \cup fA \subseteq B \cup gB \). B U. \( fB \subseteq C \cup U \). gA. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

5,4'. \( C \cup fA \subseteq B \cup gB \). A U. \( fB \subseteq C \cup U \). gB. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

5,5'. \( C \cup fA \subseteq B \cup gB \). A U. \( fB \subseteq C \cup U \). gC. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

5,6'. \( C \cup fA \subseteq B \cup gB \). A U. \( fB \subseteq C \cup U \). gA. \( \neg \)INF. \( \neg \)AL.
\( \neg \)ALF. \( \neg \)FIN. \( \neg \)NON.

LEMMA 3.12.19. 5,1', 5,2', 5,3' have \( \neg \)NON.

Proof: By Lemma 3.12.2. QED

The following pertains to 5,4'.

LEMMA 3.12.20. \( C \cup fA \subseteq B \cup gB \). A U. \( fB \subseteq C \cup U \). gB has
\( \neg \)NON.

Proof: Define \( f, g \in ELG \) as follows. For all \( n < m \), let
\( f(n,n) = 2n+2 \), \( f(n,m) = f(m,n) = 2m+1 \), \( g(n) = 4n+5 \). Let \( C \cup fA \subseteq B \cup gB \). A U. \( fB \subseteq C \cup U \). gB, where \( A,B,C \) are
nonempty.

Let \( n \in A \). Then \( n \in C \cup gB \).
case 1. \( n \in C \setminus gB \). Then \( n \in B, 2n+2 \in fB, 2n+2 \in C, 2n+2 \in fA \). This contradicts \( C \cap fA = \emptyset \).

case 2. \( n \in gB \). Let \( n = 4m+5, m \in B \). Then \( 2m+2 \in fB, 2m+2 \in C, 2m+2 \in B \). Since \( m < 2m+2 \) are from \( B \), we have \( 4m+5 \in fB \). Since \( 4m+5 = n \in A \), this contradicts \( A \cap fB = \emptyset \). QED

The following pertains to 5,6'.

**LEMMA 3.12.21.** \( C \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gA \) has \( \neg \text{NON} \).

Proof: Define \( f,g \in ELG \) as follows. For all \( n < m \), let \( f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m+1, g(n) = 4n+5 \). Let \( C \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gA \), where \( A,B,C \) are nonempty.

Let \( n = \min(A) \). Then \( n \in C \cup gA \). Clearly \( n \notin gA, n \in C, n \in B \cup gB \).

case 1. \( n \in B \). Then \( 2n+2 \in fB, 2n+2 \in C, 2n+2 \in fA \). This contradicts \( C \cap fA = \emptyset \).

case 2. \( n \in gB \). Let \( n = 4m+5, m \in B \). Then \( 2m+2 \in fB, 2m+2 \in C, 2m+2 \in B \). Since \( m < 2m+2 \) are from \( B \), we have \( 4m+5 \in fB \). Since \( 4m+5 \in A \), this contradicts \( A \cap fB = \emptyset \). QED

The following pertains to 5,5'.

**LEMMA 3.12.22.** \( C \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gC \) has \( \neg \text{NON} \).

Proof: Define \( f,g \in ELG \) as follows. For all \( n < m \), let \( f(n,n) = 2n+2, f(n,m) = 2m, f(m,n) = 4m, g(n) = 2n+1 \). Let \( C \cup fA \subseteq B \cup gB, A \cup fB \subseteq C \cup gC \), where \( A,B,C \) are nonempty.

Let \( n \in A \). Then \( 2n+2 \in fA, n \in C \cup gC \).

case 1. \( n \in C \). Then \( n \in B \cup gB \).

Suppose \( n \in B \). Then \( 2n+2 \in fB, 2n+2 \in C \). Since \( 2n+2 \in fA \), this contradicts \( C \cap fA = \emptyset \).

Suppose \( n \in gB \). Let \( n = 2m+1, m \in B \). Then \( 2m+2 \in fB, 2m+2 \in C, 2m+2 \in B \). Since \( m < 2m+2 \) are from \( B \), we have \( 4m+4 = 2n+2 \).
\( \in fB, \, 2n+2 \in C. \text{ Since } 2n+2 \in fA, \text{ this contradicts } C \cap fA = \emptyset. \)

case 2. \( n \in gC. \) Let \( n = 2m+1, \, m \in C, \, m \in B \cup gB. \)

Suppose \( m \in B. \) Then \( 2m+2 \in fB, \, 2m+2 \in C, \, 2m+2 \in B. \) Since \( m < 2m+2 \) are from \( B, \) we have \( 4m+4 = 2n+2 \in fB, \, 2n+2 \in C. \) Since \( 2n+2 \in fA, \) this contradicts \( C \cap fA = \emptyset. \)

Suppose \( m \in gB. \) Let \( m = 2r+1, \, r \in B. \) Then \( 2r+2 \in fB, \, 2r+2 \in C, \, 2r+2 \in B. \) Since \( r < 2r+2 \) are from \( B, \) we have \( 8r+8 = 4m+4 = 2n+2 \in fB, \, 2n+2 \in C. \) Since \( 2n+2 \in fA, \) this contradicts \( C \cap fA = \emptyset. \)

QED

part 6. \( C \cup fA \subseteq B \cup gC. \)

6.1'. \( C \cup fA \subseteq B \cup gC, \, B \cup fB \subseteq C \cup gU. \, gB. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

6.2'. \( C \cup fA \subseteq B \cup gC, \, B \cup fB \subseteq C \cup gU. \, gC. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

6.3'. \( C \cup fA \subseteq B \cup gC, \, B \cup fB \subseteq C \cup gC. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

6.4'. \( C \cup fA \subseteq B \cup gC, \, A \cup fB \subseteq C \cup gU. \, gB. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

6.5'. \( C \cup fA \subseteq B \cup gC, \, A \cup fB \subseteq C \cup gC. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

6.6'. \( C \cup fA \subseteq B \cup gC, \, A \cup fB \subseteq C \cup gC. \, \neg \text{INF. } \neg \text{AL. } \neg \text{ALF. } \neg \text{FIN. } \neg \text{NON. } \)

LEMMA 3.12.23. 6.1' - 6.3' have \( \neg \text{NON. } \)

Proof: By Lemma 3.12.2. QED

The following pertains to 6.5'.

LEMMA 3.12.24. \( C \cup fA \subseteq B \cup gC, \, A \cup fB \subseteq C \cup gC \) has \( \neg \text{NON. } \)

Proof: Let \( f,g \in ELG \) be defined as follows. For all \( n < m, \) let \( f(n,n) = 2n+2, \, f(n,m) = f(m,n) = 2m+1, \, g(n) = 4n+5. \) Let \( C \cup fA \subseteq B \cup gC, \, A \cup fB \subseteq C \cup gC, \) where \( A,B,C \) are nonempty.

Let \( n \in A. \) Then \( n \in C \cup gC, \, 2n+2 \in fA. \)
case 1. \( n \in C \). Then \( n \in B \cup gC, n \notin gC, n \in B, 2n+2 \in fB, 2n+2 \in C \). This contradicts \( C \cap fA = \emptyset \).

case 2. \( n \in gC \). Let \( n = 4r+5, r \in C \). Then \( r \in B \cup gC, r \in B, 2r+2 \in fB, 2r+2 \in C, 2r+2 \in B \cup gC, 2r+2 \in B \). Since \( r < 2r+2 \) are from \( B \), we have \( 4r+5 = n \in fB \). Since \( n \in A \), this contradicts \( A \cap fB = \emptyset \).

QED

The following pertains to 6, 4′.

**Lemma 3.12.25.** \( C \cup fA \subseteq B \cup gC, A \cup fB \subseteq C \cup gB \) has \( \neg \text{NON} \).

Proof: Let \( f, g \in ELG \) be defined as in the proof of Lemma 3.12.16, whose definitions we repeat here. For all \( n < m \), let \( f(2n,2n,2n) = f(2n+1,2n+1,2n+1) = 4n, f(n,m,m) = 2m, f(n,m,n) = 4m, f(m,n,n) = 8m, g(2n) = g(2n+1) = 4n+1 \). For all other triples \( a, b, c \), let \( f(a,b,c) = 2\max(a,b,c) \). Let \( C \cup fA \subseteq B \cup gC, A \cup fB \subseteq C \cup gB \), where \( A, B, C \) are nonempty.

Let \( n = \min(A) \). Then \( n \in C \cup gB \).

case 1. \( n \in C \). Then \( n \in B \cup gC \).

case 1a. \( n \in C, n \in B \). Clearly \( f(n,n,n) \in fB, f(n,n,n) \in C \). Since \( f(n,n,n) \in fA \), this contradicts \( C \cap fA = \emptyset \).

case 1b. \( n \in C, n \in gC \). Let \( n' = \min(C \cap gC) \). Let \( n' = g(m), m \in C \). Then \( m \in B \cup gC \). If \( m \in B \) then \( n' \in gB \), which contradicts \( C \cap gB = \emptyset \). Hence \( m \in gC \). So \( m \in C \cap gC \) and \( m < n' \), which is a contradiction.

case 2. \( n \in gB \). Let \( n = g(m), m \in B \). Then \( f(m,m,m) \in fB, f(m,m,m) \in fB, f(m,m,m) \in C \), \( f(m,m,m) \in B \). So \( f(f(m,m,m), f(m,m,m), f(m,m,m)) \in fB, f(f(m,m,m), f(m,m,m), f(m,m,m)) \in C \).

By the proof of Lemma 3.12.16,

\[ f(f(m,m,m), f(m,m,m), f(m,m,m)) = f(g(m), g(m), g(m)) = f(n,n,n) \in fA. \]

This contradicts \( C \cap fA = \emptyset \). QED
The following pertains to 6, 6'.

**Lemma 3.12.26.** $C \cup fA \subseteq B \cup gC$, $A \cup fB \subseteq C \cup gA$ has $\neg$NON.

**Proof:** Let $f, g \in \text{ELG}$ be defined as follows. For all $n < m$, let $f(n,n,n) = 2n$, $f(n,n,m) = 2n + 2$, $f(n,m,n) = 4m + 2$, $f(n,m,m) = 4m - 3$, $g(n) = 4n + 1$. At all other triples define $f(a,b,c) = |a,b,c| + 2$. Let $C \cup fA \subseteq B \cup gC$, $A \cup fB \subseteq C \cup gA$, where $A, B, C$ are nonempty.

Let $n = \min(A)$. We claim that $n \notin B$. To see this, let $n \in B$. Then $2n \in fB$, $2n \notin gA$, $2n \in C$, $2n \in fA$. This contradicts $C \cap fA = \emptyset$.

Since $n \in C \cup gA$, we have $n \in C$, $n \in B \cup gC$, $n \in gC$.

Let $n = 4m + 1$, $m \in C$. Suppose $m \notin gC$. Then $m \in B$, $2m \in fB$, $2m \in C$, $2m \in B$. Since $m, 2m \in B$, we have $4m + 1 \in fB$, $4m + 1 \in A$, contradicting $A \cap fB = \emptyset$. Hence $m \in gC$.

Let $p$ be greatest such that the sequence $n, g^{-1}(n), \ldots, g^{-p}(n)$ is defined and remains in $C$. Then $p \geq 2$.

Note that $g^{-p}(n) \in C \setminus gC$, $g^{-p}(n) \in B \cup gC$, $g^{-p}(n) \in B$. We have gone down by $g^{-1}$. We can go back up from $g^{-p}(n) \in B$ as follows.

First we apply the function $2n$ followed by the function $2n + 2$ (available through $f(n,n,n)$ and $f(n,n,m)$). After applying the function $2n$, we obtain an even element of $fB$, which must lie in $C, B$. After applying the function $2n + 2$, we arrive at $g^{-p+1}(n) + 1$, which is also even and lies in $C, B$. Then we apply the function $4n + 2$ successively until arriving at $g^{-1}(n) + 1$, which lies in $C, B$. Finally apply the function $4n - 3$, which arrives at $n$, and lies in $fB$. Since $n \in A$, we have contradicted $A \cap fB = \emptyset$. QED