2.5. EBRT in $A,B,f_A,f_B,\subseteq$ on (ELG, INF).

In this section, we use the tree methodology described in section 2.1 to analyze EBRT in $A,B,f_A,f_B,\subseteq$ on (ELG, INF) and (EVSD, INF). We handle both BRT settings at once, as they behave the same way for EBRT in $A,B,f_A,f_B,\subseteq$. In particular, we show that they are RCA$_0$ secure (see Definition 1.1.43).

Some of this treatment is the same as for EBRT in $A,B,f_A,f_B,\subseteq$ on (SD, INF) given in section 2.4. However, many new features appear that makes this section considerably more involved than section 2.4.

A key difference between EBRT in $A,B,f_A,f_B,\subseteq$ on (SD, INF) and on (ELG, INF) is that the Compelmentation Theorem holds on (SD, INF), yet fails on (ELG, INF). E.g., it fails for $f(x) = 2x$.

Let $f:N^k \to N$ be partial. Define the following series of sets by induction $i \geq 1$.

\[
S_1 = N.
\]
\[
S_{i+1} = N \setminus fS_i.
\]

**LEMMA 2.5.1.** $S_2 \subseteq S_4 \subseteq S_6 \subseteq \ldots \subseteq \ldots \subseteq S_5 \subseteq S_3 \subseteq S_1$. I.e., for all $i \geq 1$, $S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}$.

**Proof:** We argue by induction on $i \geq 1$. The basis case is $S_2 \subseteq S_4 \subseteq S_3 \subseteq S_1$.

To see this, clearly

\[
S_3 \subseteq S_1.
\]
\[
N \setminus S_1 \subseteq N \setminus S_3.
\]
\[
S_2 \subseteq S_4.
\]
\[
S_2 \subseteq S_1.
\]
\[
N \setminus S_1 \subseteq N \setminus S_2.
\]
\[
S_2 \subseteq S_3.
\]
\[
fS_2 \subseteq fS_3.
\]
\[
N \setminus fS_3 \subseteq N \setminus fS_2.
\]
\[
S_4 \subseteq S_3.
\]

Now assume the induction hypothesis

\[
S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}.
\]
Then

\[ fS_{2i} \subseteq fS_{2i+2} \subseteq fS_{2i+1} \subseteq fS_{2i-1}. \]
\[ N\backslash fS_{2i-1} \subseteq N\backslash fS_{2i+1} \subseteq N\backslash fS_{2i+2} \subseteq N\backslash fS_{2i}. \]
\[ S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+3} \subseteq S_{2i+1}. \]
\[ fS_{2i} \subseteq fS_{2i+2} \subseteq fS_{2i+3} \subseteq fS_{2i+1}. \]
\[ N\backslash fS_{2i+1} \subseteq N\backslash fS_{2i+2} \subseteq N\backslash fS_{2i+3} \subseteq N\backslash fS_{2i}. \]
\[ S_{2i+2} \subseteq S_{2i+4} \subseteq S_{2i+3} \subseteq S_{2i+1}. \]

QED

**Lemma 2.5.2.** Let \( f: N^k \to N \) be partial, where each \( f^{-1}(n) \) is finite. Let \( A = S_2 \cup S_4 \cup \ldots \), and \( B = S_1 \cap S_3 \cap \ldots \). Then \( A \subseteq B \), \( A = N \setminus fB \), \( B = N \setminus fA \).

**Proof:** Let \( A,B \) be as given. By Lemma 2.5.1, \( A \subseteq B \).

Fix \( i \geq 1 \). \( S_{2i} = N \setminus fS_{2i-1} \), \( S_{2i} \cap fS_{2i-1} = \emptyset \), \( S_{2i} \cap fB = \emptyset \).

Since \( i \geq 1 \) is arbitrary, \( A \cap fB = \emptyset \). I.e., \( A \subseteq N \setminus fB \).

Since \( S_{2i+1} = N \setminus fS_{2i} \), we see that for all \( j \geq i \), \( S_{2i+1} \cap fS_{2j} = \emptyset \). Hence \( S_{2i+1} \cap fA = \emptyset \). Since \( i \geq 1 \) is arbitrary, \( B \cap fA = \emptyset \). I.e., \( B \subseteq N \setminus fA \).

Now let \( n \in N \setminus fB \). We claim that for some \( j \geq 0 \), \( n \notin fS_{2j+1} \). Suppose that for all \( j \geq 0 \), \( n \in fS_{2j+1} \). Since \( f^{-1}(n) \) is finite, there exists \( x \in f^{-1}(n) \) which lies in infinitely many \( S_{2j+1} \). Hence there exists \( x \in f^{-1}(n) \) such that \( x \in B \). Therefore \( n \in fB \). This establishes the claim. Fix \( j \geq 0 \) such that \( n \notin fS_{2j+1} \). Then \( n \in S_{2j+2} \), and so \( n \in A \). This establishes that \( A = N \setminus fB \).

Finally, let \( n \in N \setminus fA \). Then for all \( i \), \( n \notin fS_{2i} \). Hence for all \( j \), \( n \in S_{2j+1} \). Therefore \( n \in B \). This establishes that \( B = N \setminus fA \). QED

**Lemma 2.5.3.** Let \( f: [0,n]^k \to [0,n] \) be partial, \( n \geq 0 \). There exist \( A \subseteq B \subseteq [0,n] \) such that \( A = [0,n] \setminus fB \) and \( B = [0,n] \setminus fA \).

**Proof:** Let \( n,f \) be as given. Obviously \( f:N^k \to N \) is partial, and each \( f^{-1}(n) \) is finite. By Lemma 2.5.2, let \( A = S_2 \cup S_4 \cup \ldots \), and \( B = S_1 \cap S_3 \cap \ldots \). Then \( A \subseteq B \), \( A = N \setminus fB \), \( B = N \setminus fA \). Note that \( A \cap [0,n] \subseteq B \cap [0,n] \), \( A \cap [0,n] = [0,n] \setminus fB \), \( B \cap [0,n] \setminus fA \). QED
LEMMA 2.5.4. For all \( f \in \text{EVSD} \) there exist infinite \( A \subseteq B \subseteq \mathbb{N} \) such that \( B \cup fA = A \cup fB = \mathbb{N} \).

Proof: Let \( f \in \text{EVSD} \). Let \( n \geq 1 \) be such that \(|x| \geq n \rightarrow f(x) > |x|\). Let \( f' \) be the restriction of \( f \) to those elements of \([0,n-1]^k\) whose value lies in \([0,n-1]\). Then \( f':[0,n-1]^k \rightarrow [0,n-1] \) is partial.

By Lemma 2.5.3, let \( A' \subseteq B' \subseteq [0,n-1] \), where \( A' = [0,n-1] \setminus f'B' \) and \( B' = [0,n-1] \setminus f'A' \).

We now define the required \( A, B \) by induction. Membership in \( A, B \) for \( m < n \) is just membership in \( A', B' \). Thus for all \( m < n \),

\[
\begin{align*}
m \in B & \iff m \in B' \iff m \notin f'A' \iff m \notin fA. \\
m \in A & \iff m \in A' \iff m \notin f'B' \iff m \notin fB.
\end{align*}
\]

Now suppose membership in \( A, B \) has been defined for all \( 0 \leq i < m \), where \( m \geq n \), and we have \( A \subseteq B \) thus far.

case 1. \( m \notin fA \) thus far. Put \( m \in A, B \).

case 2. \( m \in fA \) thus far. Put \( m \notin A, B \).

This defines membership of \( m \) in \( A, B \). Note that we still have \( A \subseteq B \).

Now let \( A, B \) be the result of this inductive construction. Note that by the choice of \( n \), all of the “thus far” remain true of the actual \( A, B \), where \( m \geq n \). Thus we have for all \( m \geq n \),

\[
\begin{align*}
A \subseteq B. \\
m \notin fA & \iff m \in A \iff m \in B. \\
m \notin A & \rightarrow m \in fA \rightarrow m \in fB.
\end{align*}
\]

Hence for all \( m \geq n \), \( m \in B \cup fA \) and \( m \in A \cup fB \). Since this also holds for \( m < n \), this holds for all \( m \in \mathbb{N} \).

Finally, suppose \( A \) is finite. Then \( fA \) is finite, and so eventually all \( m \) are placed in \( A \). Thus \( A \) is infinite. Hence \( A \) is infinite. QED

LEMMA 2.5.5. For all \( f \in \text{EVSD} \) there exist infinite \( A \subseteq B \subseteq \mathbb{N} \) such that \( A \cup fB = \mathbb{N} \) and \( B \cap fA = \emptyset \).
Proof: Let \( f \in \text{EVSD} \). Let \( n, A', B' \) be as in the first paragraph of the proof of Lemma 2.5.4.

We now define the required \( A, B \) by induction. Membership in \( A, B \) for \( m < n \) is just membership in \( A', B' \). Thus for all \( m < n \),

\[
\begin{align*}
\text{if } m \in B & \iff m \in B' \iff m \notin f'A' \iff m \notin fA, \\
\text{if } m \in A & \iff m \in A' \iff m \notin f'B' \iff m \notin fB.
\end{align*}
\]

Now suppose membership in \( A, B \) has been defined for all \( i < m \), where \( m \geq n \), and we have \( A \subseteq B \) thus far.

Case 1. \( m \notin fB \) thus far. Put \( m \in A, B \).

Case 2. \( m \in fB \) thus far. Put \( m \notin A, B \).

This defines membership of \( m \) in \( A, B \). Note that we still have \( A \subseteq B \).

Now let \( A, B \) be the result of this inductive construction. Note that by the choice of \( n \), all of the “thus far” remain true of the actual \( A, B \), where \( m \geq n \). Thus we have for all \( m \geq n \),

\[
A \subseteq B.
\]

\[
\begin{align*}
\text{if } m \notin fB & \iff m \in A \iff m \in B, \\
\text{if } m \in B & \Rightarrow m \notin fB \Rightarrow m \notin fA.
\end{align*}
\]

Hence for all \( m \geq n \), \( m \in A \cup fB \) and \( m \notin B \cap fA \). Since this also holds for \( m < n \), this holds for all \( m \in \mathbb{N} \).

Finally, suppose \( A \) is finite. Then eventually all \( m \) are placed in \( fB \). Hence eventually all \( m \) are placed outside \( B \). Hence \( B \) is finite. So \( fB \) is finite. Then eventually all \( m \) are put in \( A, B \). This is a contradiction. QED

**Lemma 2.5.6.** There exists \( f \in \text{ELG} \) such that \( f^{-1}(0) = \{(0,\ldots,0)\} \), \( f(\mathbb{N}\setminus\{0\}) \subseteq 2\mathbb{N} + 1 \), and for all \( A \subseteq \mathbb{N} \) containing \( 0 \), \( fA \cap 2\mathbb{N} \subseteq A \rightarrow fA \) is cofinite.

Proof: Let \( g \in \text{ELG} \cap \text{SD} \) be given by Lemma 3.2.1. We define 4-ary \( f \in \text{ELG} \) as follows. \( f(0,0,0,0) = 0 \). \( f(0,n,m,r) = g(n,m,r) \) if \( (n,m,r) \neq (0,0,0) \). \( f(t,n,m,r) = 2|t,n,m,r| + 1 \) if \( t \neq 0 \). Obviously \( f \in \text{ELG} \cap \text{SD} \), \( f(\mathbb{N}\setminus\{0\}) \subseteq 2\mathbb{N} + 1 \), and \( f^{-1}(0) = \{(0,0,0,0)\} \).
Now let $A \subseteq N$, $0 \in A$, where $fA \cap 2N \subseteq A$. Since $gA \subseteq fA$, we have $gA \cap 2N \subseteq A$, and so by Lemma 3.2.1, $gA$ is cofinite. Hence $fA$ is cofinite. QED

**Lemma 2.5.7.** The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fB = \emptyset$, $B \cup fB = N$, and $fB \subseteq B \cup fA$.

Proof: Let $f \in ELG$ be given by Lemma 2.5.6. Let $A \cap fB = \emptyset$, $B \cup fB = N$, and $fB \subseteq B \cup fA$, where $A$ is infinite. Now $0 \in B \lor 0 \in fB$. Since $f^{-1}(0) = \{(0,0,0,0)\}$, we have $0 \in B$, $0 \in fB$, $0 \notin A$. Therefore $fA \subseteq 2N+1$. Since $fB \subseteq B \cup fA$, we have $fB \cap 2N \subseteq B$. Therefore $fB$ is cofinite. This contradicts $A \cap fB = \emptyset$. QED

**Lemma 2.5.8.** The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $B \cup fA = N$ and $A \cap fB = \emptyset$.

Proof: Let $f$ be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq N$, $B \cup fA = N$, and $A \cap fB = \emptyset$, where $A$ is infinite. Since $0 \in B \lor 0 \in fA$, we have $0 \in B \lor 0 \in fA$. If $0 \in fA$ then $0 \in A,B$, because $f^{-1}(0) = \{(0,0,0,0)\}$. Hence $0 \notin fA$, $0 \notin A$. Therefore $fA \subseteq 2N+1$. Since $B \cup fA = N$, we have $2N \subseteq B$. By Lemma 3.2.1, $fB$ is cofinite. By $A \cap fB = \emptyset$, $A$ is finite. But $A$ is infinite. QED

**Lemma 2.5.9.** For all $f \in EVSD$ there exist infinite $A \subseteq B \subseteq N$ such that $B \cup fA = N$ and $A \subseteq fB$.

Proof: Let $n$ be such that $|x| \geq n \rightarrow f(x) > |x|$. We can use Lemma 2.4.1 with $N$ replaced by $[n,\infty)$. Let $A,B \subseteq [n,\infty)$, $A \subseteq B$, $B \cup fA = [n,\infty)$ and $A = B \cap fB$, where $A$ is infinite. Then $B \cup fA = [n,\infty)$, $A \subseteq fB$. Replace $B$ with $B \cup [0,n-1]$. QED

**Lemma 2.5.10.** The following is false. For all $f \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that $A \cap fA = \emptyset$, $B \cup fB = N$, $B \cap fB \subseteq A \cup fA$.

Proof: Let $f$ be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq N$ such that $A \cap fA = \emptyset$, $B \cup fB = N$, $B \cap fB \subseteq A \cup fA$, where $A,B$ are infinite. Then $0 \in B \cup fB$, and so $0 \in B \cap fB$. Hence $0 \in A \cup fA$, in which case $0 \in A \cap fA$. QED
LEMMA 2.5.11. For all \( f \in \text{EVSD} \) there exist infinite \( A \subseteq B \subseteq N \) such that \( A \cup fB = N \) and \( fA \subseteq B \).

Proof: Let \( f' \) be the restriction of \( f \) to \( \{x: f(x) > |x|\} \). Then \( f' \) is defined at all but finitely many elements of \( \text{dom}(f) \). As remarked right after Lemma 2.4.5, Lemma 2.4.2 holds even for partial functions, and so in particular for \( f' \). Let \( A \subseteq B \subseteq N \), where \( A \cup fB = N \) and \( fA \subseteq B \) and \( A \) is infinite. Let \( A' = N \setminus fB \subseteq A \). Since \( f'B \) contains all but finitely many elements of \( fB \), we see that \( A' \) remains infinite. Then \( A', B \) are as required. QED

LEMMA 2.5.12. Let \( f \in \text{EVSD} \). There exist infinite \( A \subseteq B \subseteq N \) such that \( fB \subseteq B \cup fA \) and \( A = B \cap fB \).

Proof: Let \( n \) be such that \( |x| \geq n \rightarrow f(x) > |x| \). We can use Lemma 2.4.1 with \( N \) replaced by \( [n, \infty) \). Let \( A, B \subseteq [n, \infty) \), \( A \subseteq B, B \cup fA = [n, \infty) \), and \( A = B \cap fB \), where \( A \) is infinite. Since \( fB \subseteq [n, \infty) \), the proof is complete. QED

LEMMA 2.5.13. Let \( f \in \text{EVSD} \). There exist infinite \( A \subseteq N \) such that \( A \cap f(A \cup fA) = \emptyset \).

Proof: Let \( n \) be such that \( |x| \geq n \rightarrow f(x) > |x| \). Define \( n_0 < n_1 < \ldots \) by induction as follows. Let \( n_0 = n \). Suppose \( n_i \) has been defined, \( i \geq 0 \). Let \( n_{i+1} \) be greater than all elements of \( f(A \cup fA) \), thus far. Finally, let \( A = \{n_0, n_1, \ldots\} \). QED

LEMMA 2.5.14. Let \( f \in \text{EVSD} \) and let \( X \subseteq N \), where \( \min(X) \) is sufficiently large. There exists a unique \( A \) such that \( A \subseteq X \subseteq A \cup fA \). If \( X \) is infinite then \( A \) is infinite.

Proof: Let \( f, X \) be as given. Then \( |x| \geq \min(X) \rightarrow f(x) > |x| \). We can use Lemma 2.4.3 with \( N \) replaced by \( [\min(X), \infty) \). Let \( A \subseteq X \cap [\min(X), \infty) \subseteq A \cup fA \).

For uniqueness, suppose \( A \subseteq X \subseteq A \cup fA, A' \subseteq X \subseteq A' \cup fA' \), and let \( n = \min(A \Delta A') \). Since \( f \in \text{SD} \), clearly \( n \in fA \leftrightarrow n \in fA' \). This is a contradiction. QED

As in section 2.4, we start with the 9 elementary inclusions in \( A, B, fA, fB, \subseteq \).

EBRT in \( A, B, fA, fB, \subseteq \) on \( (\text{ELG, INF}), (\text{EVSD, INF}) \).

\( A \cap fA = \emptyset \).
\( B \cup fB = N \).
Our classification amounts to a determination of the subsets $S$ of the above nine inclusions for which

$$(\forall f \in \text{ELG}) (\exists A \subseteq B \text{ from INF}) (S)$$

$$(\forall f \in \text{EVSD}) (\exists A \subseteq B \text{ from INF}) (S)$$

holds, where $S$ is interpreted conjunctively.

$\text{EBRT}$ in $A,B,fA,fB, \subseteq$ on $(\text{ELG,INF}), (\text{EGS} \cap \text{SD,INF}).$*

# 5

$A \cap fA = \emptyset.$
$B \cup fB = N.$
$fA \subseteq B.$
$A \subseteq fB.$
$B \subseteq A \cup fB.$
$fB \subseteq B \cup fA.$
$A \cap fB \subseteq fA.$
$A \cap fB = \emptyset.$
$B \cap fA \subseteq A.$
$B \cap fA = \emptyset.$
$B \cap fB \subseteq A \cup fA.$

LIST 1.

$A \cap fA = \emptyset:$
$B \cup fB = N.$
$fA \subseteq B.$
$A \subseteq fB.$
$B \subseteq A \cup fB.$
$fB \subseteq B \cup fA.$
$A \cap fB \subseteq fA.$
$A \cap fB = \emptyset.$
$B \cap fA \subseteq A.$
$B \cap fA = \emptyset.$
$B \cap fB \subseteq A \cup fA.$

LIST 1*.

# 6

$A \cap fA = \emptyset:$
$B \cap fA = \emptyset.$
$A \cap fB = \emptyset.$
\[ fA \subseteq B. \]
\[ A \subseteq fB. \]
\[ B \cup fB = N. \]
\[ B \subseteq A \cup fB. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A \cup fA. \]

**LIST 1.1.**

\[ A \cap fA = \emptyset: \text{Redundant.} \]
\[ B \cap fA = \emptyset: \]
\[ A \cap fB = \emptyset. \]
\[ fA \subseteq B. \text{No.} \]
\[ A \subseteq fB. \]
\[ B \cup fB = N. \]
\[ B \subseteq A \cup fB. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A. \]

**LIST 1.1.*

\[ \# 4 \]

\[ B \cap fA = \emptyset: \]
\[ A \cap fB = \emptyset. \]
\[ A \subseteq fB. \]
\[ B \cup fB = N. \]
\[ B \subseteq A \cup fB. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A. \]

**LIST 1.1.1.**

\[ B \cap fA = \emptyset: \]
\[ A \cap fB = \emptyset: \]
\[ A \subseteq fB. \text{No.} \]
\[ B \cup fB = N. \]
\[ B \subseteq A \cup fB. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A. \]
\[ B \cap fB = \emptyset. \]

**LIST 1.1.1.*

\[ \# 2 \]

\[ B \cap fA = \emptyset: \]
\[ A \cap fB = \emptyset: \]
\[ B \cup fB = N. \]
\[ B \subseteq A \cup fB. \]
\( \mathbb{B} \subseteq \mathbb{B} \cup fA. \)

\( B \cap fB = \emptyset. \)

LIST 1.1.1.1.

\( B \cap fA = \emptyset: \)
\( A \cap fB = \emptyset: \)
\( B \cup fB = N: \)
\( fB \subseteq A \cup fB. \) A \cup fB = N.
\( fB \subseteq B \cup fA. \) B \cup fA = N. No. Lemma 2.5.8.
\( B \cap fB = \emptyset. \) No. Lemma 2.5.10.

LIST 1.1.1.1.*

# 0

\( B \cap fA = \emptyset: \)
\( A \cap fB = \emptyset: \)
\( B \cup fB = N: \)
\( A \cup fB = N. \)

Entirely RCA\textsubscript{0} correct. Lemma 2.5.5.

LIST 1.1.1.2.

\( B \cap fA = \emptyset: \)
\( A \cap fB = \emptyset: \)
\( B \subseteq A \cup fB: \)
\( fB \subseteq B \cup fA. \)
\( B \cap fB = \emptyset. \)

Entirely RCA\textsubscript{0} correct. Set \( A \cap fA = \emptyset, B = A. \)

LIST 1.1.2.

\( B \cap fA = \emptyset: \)
\( A \subseteq fB: \)
\( B \cup fB = N. \)
\( B \subseteq A \cup fB. \) B \subseteq fB. No.
\( fB \subseteq B \cup fA. \)
\( B \cap fB \subseteq A. \)

LIST 1.1.2.*

# 2

\( B \cap fA = \emptyset: \)
\( A \subseteq fB: \)
\( B \cup fB = N. \)
$fB \subseteq B \cup fA.$
$B \cap fB \subseteq A.$

**LIST 1.1.2.1.**

$B \cap fA = \emptyset:$
$A \subseteq fB:$
$B \cup fB = N:$
$fB \subseteq B \cup fA. B \cup fA = N.$
$B \cap fB \subseteq A.$ No. Lemma 2.5.10.

**LIST 1.1.2.1.*

# 0

$B \cap fA = \emptyset:$
$A \subseteq fB:$
$B \cup fB = N:$
$B \cup fA = N.$

Entirely RCA\textsubscript{0} correct. Lemma 2.5.9.

**LIST 1.1.2.2.**

$B \cap fA = \emptyset:$
$A \subseteq fB:$
$fB \subseteq B \cup fA:$
$B \cap fB \subseteq A.$

Entirely RCA\textsubscript{0} correct. Lemma 2.5.12.

**LIST 1.1.3.**

$B \cap fA = \emptyset:$
$B \cup fB = N:$
$B \subseteq A \cup fB. A \cup fB = N.$
$fB \subseteq B \cup fA. B \cup fA = N.$
$B \cap fB \subseteq A.$ No. Lemma 2.5.10.

**LIST 1.1.3.*

# 0

$B \cap fA = \emptyset:$
$B \cup fB = N:$
$A \cup fB = N.$
$B \cup fA = N.$

Entirely RCA\textsubscript{0} correct. Lemma 2.5.4.
LIST 1.1.4.

\[ B \cap fA = \emptyset: \]
\[ B \subseteq A \cup fB: \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq fA. \]

Entirely RCA_0 correct. Set \( A \cap fA = \emptyset, B = A. \)

LIST 1.2.

A \cap fA = \emptyset: Redundant.
A \cap fB = \emptyset:
\[ fA \subseteq B. \]
A \subseteq fB. No.
B \cup fB = N.
B \subseteq A \cup fB.
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A \cup fA. \]
\[ B \cap fB \subseteq fA. \]

LIST 1.2.*

# 3

A \cap fB = \emptyset:
\[ fA \subseteq B. \]
B \cup fB = N.
B \subseteq A \cup fB.
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq fA. \]

LIST 1.2.1.

A \cap fB = \emptyset:
\[ fA \subseteq B: \]
B \cup fB = N.
B \subseteq A \cup fB.
\[ fB \subseteq B \cup fA. \]
\[ fB \subseteq B. \]
No. Lemma 2.4.4.
B \cap fB \subseteq fA.

LIST 1.2.1.*

# 2

A \cap fB = \emptyset:
\[ fA \subseteq B: \]
B \cup fB = N.
B \subseteq A \cup fB.
B ∩ fB ⊆ fA.

LIST 1.2.1.1.

A ∩ fB = Ø:
  fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB.
B ∩ fB ⊆ fA. No. Lemma 2.5.10.

LIST 1.2.1.1.*
  # 0

A ∩ fB = Ø:
  fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB.

Entirely RCA₀ correct. See Lemma 2.5.11.

LIST 1.2.1.2.
  # 0

A ∩ fB = Ø:
  fA ⊆ B:
B ⊆ A ∪ fB:
B ∩ fB ⊆ fA.

Entirely RCA₀ correct. Le A be given by Lemma 2.5.13. Set B = A ∪ fA.

LIST 1.2.2.

A ∩ fB = Ø:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. No. Lemma 2.5.7.
B ∩ fB ⊆ fA. No. Lemma 2.5.10.

LIST 1.2.2.*
  # 0

A ∩ fB = Ø:
B ∪ fB = N:
A ∪ fB = N.

Entirely RCA₀ correct. Lemma 2.5.5.
LIST 1.2.3.

A ∩ fB = ∅:
B ⊆ A ∪ fB:
 fB ⊆ B ∪ fA.
B ∩ fB ⊆ fA.

Entirely RCA0 correct. Set A ∩ fA = ∅, B = A.

LIST 1.3.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.*

# 3

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB.
B ∪ fB = N.
B ⊆ A ∪ fB.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N.
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.*

# 2

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N.
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA.

LIST 1.3.1.1.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.3.1.1.*
# 0

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
B ∪ fB = N:
fB ⊆ B.

Entirely RCA₀ correct. Let A be given by Lemma 2.4.3 with A ⊆ fN ⊆ A ∪ fA. Set B = N.

LIST 1.3.1.2.

A ∩ fA = ∅:
fA ⊆ B:
A ⊆ fB:
fB ⊆ B:
B ∩ fB ⊆ A ∪ fA.

Entirely RCA₀ correct. Let B = [n, ∞), n sufficiently large. By Lemma 2.5.14, let A ⊆ fB ⊆ A ∪ fA.

LIST 1.3.2.

A ∩ fA = ∅:
fA ⊆ B:
B ∪ fB = N:
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B. B = N.
B ∩ fB ⊆ A ∪ fA. No. Lemma 2.5.10.

LIST 1.3.2.*
# 0
\[ A \cap fA = \emptyset: \]
\[ fA \subseteq B: \]
\[ B \cup fB = N: \]
\[ A \cup fB = N. \]
\[ B = N. \]

Entirely RCA\(_0\) correct. Set \( A \neq N\backslash fN, B = N. \)

LIST 1.3.3.

\[ A \cap fA = \emptyset: \]
\[ fA \subseteq B: \]
\[ fB \subseteq B. \]
\[ B \cap fB \subseteq A \cup fA. \]

Entirely RCA\(_0\) correct. Let \( B = [n, \infty) \) for \( n \) sufficiently large. Let \( A \subseteq B \subseteq A \cup fA. \) by Lemma 2.5.14.

LIST 1.4.

\[ A \cap fA = \emptyset: \]
\[ A \subseteq fB: \]
\[ B \cup fB = N. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A \cup fA. \]

LIST 1.4.*

\# 2

\[ A \cap fA = \emptyset: \]
\[ A \subseteq fB: \]
\[ B \cup fB = N. \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A \cup fA. \]

LIST 1.4.1.

\[ A \cap fA = \emptyset: \]
\[ A \subseteq fB: \]
\[ B \cup fB = N: \]
\[ fB \subseteq B \cup fA. \]
\[ B \cap fB \subseteq A \cup fA. \] No. Lemma 2.5.10.

LIST 1.4.1.*
Entirely RCA\(_0\) correct. Let \(A \subseteq fN \subseteq A \cup fA\) be given by Lemma 2.4.3. Set \(B = N\).

LIST 1.4.2.

\[
\begin{align*}
A \cap fA &= \emptyset, \\
A &\subseteq fB, \\
B \cup fB &= N, \\
fB &\subseteq B \cup fA.
\end{align*}
\]

Entirely RCA\(_0\) correct. Lemma 2.5.12.

LIST 1.5.

\[
\begin{align*}
A \cap fA &= \emptyset, \\
B \cup fB &= N, \\
B &\subseteq A \cup fB, \\
fB &\subseteq B \cup fA, \\
B \cap fB &\subseteq A \cup fA. 
\end{align*}
\]

Entirely RCA\(_0\) correct. No. Lemma 2.5.10.

LIST 1.5.*

\[
\begin{align*}
A \cap fA &= \emptyset, \\
B \cup fB &= N, \\
B &\subseteq A \cup fB, \\
fB &\subseteq B \cup fA.
\end{align*}
\]

Entirely RCA\(_0\) correct. Lemma 2.5.4.

LIST 1.6.

\[
\begin{align*}
A \cap fA &= \emptyset, \\
B &\subseteq A \cup fB, \\
fB &\subseteq B \cup fA, \\
B \cap fB &\subseteq A \cup fA.
\end{align*}
\]

Entirely RCA\(_0\) correct. Let \(A \cap fA = \emptyset, B = A\).

LIST 2.
B ∪ fB = N:
fA ⊆ B.
A ⊆ fB.
B ⊆ A ∪ fB. A ∪ fB = N.
fB ⊆ B ∪ fA. B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.*
# 3

B ∪ fB = N:
fA ⊆ B.
A ⊆ fB.
A ∪ fB = N.
B ∪ fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB.
A ∪ fB = N.
B ∪ fA = N. B = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A. fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.*
# 2

B ∪ fB = N:
fA ⊆ B:
A ⊆ fB.
A ∪ fB = N.
B = N.
A ∩ fB ⊆ fA.
fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 2.1.1.
B \cup fB = N:
fa \subseteq B:
A \subseteq fB:
A \cup fB = N. fb = N. No. Lemma 2.4.5.
B = N.
A \cap fB \subseteq fa. A \subseteq fa. No. Lemma 2.4.5.
fa \subseteq A.
B \cap fB \subseteq A \cup fa.

LIST 2.1.1.*
# 0

B \cup fB = N:
fa \subseteq B:
A \subseteq fB:
B = N.
fa \subseteq A.
B \cap fB \subseteq A \cup fa.

Entirely RCA_0 correct. Set A = fN, B = N.

LIST 2.1.2.

B \cup fB = N:
fa \subseteq B:
A \subseteq fB:
A \cup fB = N.
B = N.
A \cap fB \subseteq fa.
B \cap fa \subseteq A.
B \cap fB \subseteq A \cup fa.

Entirely RCA_0 correct. Set A = B = N.

LIST 2.2.

B \cup fB = N:
A \subseteq fB:
A \cup fB = N. Yes.
B \cup fa = N.
A \cap fb \subseteq fa. A \subseteq fa. No. Lemma 2.4.5.
B \cap fa \subseteq A.
B \cap fB \subseteq A \cup fa.

LIST 2.2.*
# 0

B \cup fB = N:
A ⊆ fB:
fB ⊆ B U fA. B U fA = N.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A U fA.

Entirely RCA₀ correct. Set A = fN, B = N.

LIST 2.3.

B U fB = N:
A U fB = N.
B U fA = N.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A U fA.

Entirely RCA₀ correct. Set A = B = N.

LIST 3.

fA ⊆ B:
A ⊆ fB.
B ⊆ A U fB.
fB ⊆ B U fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A U fA.

LIST 3*.

# 2

fA ⊆ B:
A ⊆ fB.
B ⊆ A U fB.
fB ⊆ B U fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A U fA.

LIST 3.1.

fA ⊆ B:
A ⊆ fB:
B ⊆ A U fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B U fA.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 3.1.*
# 0

fA ⊆ B:
A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA₀ correct. Set A = fN, B = N.

LIST 3.2.

fA ⊆ B:
B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA₀ correct. Set A = B = fN.

LIST 4.

A ⊆ fB:
B ⊆ A ∪ fB. B ⊆ fB. No. Lemma 2.4.5.
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA. A ⊆ fA. No. Lemma 2.4.5.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

LIST 4.*
# 0

A ⊆ fB:
fB ⊆ B ∪ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA₀ correct. Set A = fN, B = N.

LIST 5.

B ⊆ A ∪ fB:
fB ⊆ B ∪ fA.
A ∩ fB ⊆ fA.
B ∩ fA ⊆ A.
B ∩ fB ⊆ A ∪ fA.

Entirely RCA₀ correct. Set A = B = N.

THEOREM 2.5.15. EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) have the same correct formats. EBRT in A,B,fA,fB,⊆ on (ELG,INF) and (EVSD,INF) are RCA₀ secure.

Proof: We have presented an RCA₀ classification of EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) in the sense of the tree methodology of section 2.1. All of the documentation works equally well on (ELG,INF) and (EVSD,INF). We have stayed within RCA₀. QED

THEOREM 2.5.16. There are at most 26 maximal α correct α formats, where α is EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF).

Proof: Here is the list of numerical labels of terminal vertices in the RCA₀ classification of EBRT in A,B,fA,fB,⊆ on (ELG,INF), (EVSD,INF) given above:

1.1.1.1.*
1.1.1.2.
1.1.2.1.*
1.1.2.2.
1.1.3.*
1.1.3.
1.2.1.1.*
1.2.1.2.
1.2.2.*
1.2.3.
1.3.1.1.*
1.3.1.2.
1.3.2.*
1.3.3.
1.4.1.*
1.4.2.
1.5.*
1.6.
2.1.1.*
2.1.2.
2.2.*
2.3.
3.1.*
3.2.
4.*
5.
The count is 26. Apply Theorem 2.1.5. QED